JACOBI-LIKE STRUCTURES IN THE LINE BUNDLE LANGUAGE

Eugen-Mihaita CIOROIANU

University of Craiova

cioroianu.eugen@ucv.ro

Belgrade, September 8, 2021

Outline

- Motivation
- Trivial line bundle formulation
- Non-trivial line bundle formulation
- References

Outline

- Motivation
- Trivial line bundle formulation
- Non-trivial line bundle formulation
- References

Outline

- Motivation
- Trivial line bundle formulation
- Non-trivial line bundle formulation
- References

Outline

- Motivation
- Trivial line bundle formulation
- Non-trivial line bundle formulation
- References

Outline

- Motivation
- Trivial line bundle formulation
- Non-trivial line bundle formulation
- References

Motivation

$$\text{P/P-like} \longrightarrow \text{J/J-like} \longrightarrow \text{`homogeneous' P/P-like}$$

$$(M,\Pi) \longrightarrow (M,\Pi,E) \longrightarrow (\tilde{M},\tilde{\Pi})$$

$$(M,\Pi,\mathrm{d}\omega) \longrightarrow (M,\Pi,E,\omega) \longrightarrow (\tilde{M},\tilde{\Pi},\tilde{\mathrm{d}}\tilde{\omega})$$

$$(M,\Pi,\phi) \longrightarrow (M,\Pi,E,\phi,\omega) \longrightarrow (\tilde{M},\tilde{\Pi},\tilde{\phi})$$

Outline

- Motivation
- Trivial line bundle formulation
- Non-trivial line bundle formulation
- References

Trivial line bundle formulation

- From Jacobi pairs to twisted Jacobi pairs
- Relaxing twisted Jacobi pairs: Jacobi pair with background
- Jacobi-like pairs as distinguished elements of a Lie algebroid

Trivial line bundle formulation

- From Jacobi pairs to twisted Jacobi pairs
- Relaxing twisted Jacobi pairs: Jacobi pair with background
- Jacobi-like pairs as distinguished elements of a Lie algebroid

Trivial line bundle formulation

- From Jacobi pairs to twisted Jacobi pairs
- Relaxing twisted Jacobi pairs: Jacobi pair with background
- Jacobi-like pairs as distinguished elements of a Lie algebroid

Trivial line bundle formulation

- From Jacobi pairs to twisted Jacobi pairs
- Relaxing twisted Jacobi pairs: Jacobi pair with background
- Jacobi-like pairs as distinguished elements of a Lie algebroid

From Jacobi pairs to twisted Jacobi pairs

Let M be a smooth manifold. By definition, a Jacobi pair (Π, E) consists in

$$\Pi \in \mathfrak{X}^2(M), \quad E \in \mathfrak{X}^1(M)$$

that enjoy the properties

$$[\Pi, \Pi] + 2\Pi \wedge E = 0, \quad [\Pi, E] = 0, \tag{1}$$

with $[\bullet, \bullet]$ the Schouten-Nijenhuis bracket in the Gerstenhaber algebra of multi-vector fields

$$\mathfrak{X}^\bullet(M) \equiv \mathcal{F}(M) \oplus \mathfrak{X}^1(M) \oplus \cdots \oplus \mathfrak{X}^{\dim M}(M)$$

coming from the Lie algebra of smooth vector fields

$$\left(\mathfrak{X}(M) \equiv \mathfrak{X}^1(M), [\bullet, \bullet]\right).$$

From Jacobi pairs to twisted Jacobi pairs

A Jacobi pair, naturally structures the vector space $\mathcal{F}(M)$ as a Lie algebra, but not a Poisson one with respect to

$$\begin{aligned} &\{\bullet, \bullet\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M), \\ &\{f, g\} \equiv i_{\Pi} \mathrm{d}f \wedge \mathrm{d}g + i_E (f \mathrm{d}g - g \mathrm{d}f). \end{aligned} \tag{2}$$

The bracket exhibits the 'Hamiltonian' morphism of Lie algebras

$$\begin{aligned} &\mathcal{H} : \mathcal{F}(M) \to \mathfrak{X}^1(M), \\ &\mathcal{H}(f) \equiv X_f = \Pi^\sharp \mathrm{d}f + fE, \end{aligned} \tag{3}$$

with

$$\Pi^\sharp : T^*M \to TM, \quad \Pi^\sharp \alpha \equiv -j_\alpha \Pi. \tag{4}$$

The Hamiltonian vector fields enjoy the properties

$$[X_f, X_g] = X_{\{f,g\}}, \quad [X_f, E] = -X_{\mathcal{L}_E f} \tag{5}$$

From Jacobi pairs to twisted Jacobi pairs

A Jacobi pair is said to be transitive if the 'Hamiltonian' distribution coincides with the tangent one i.e iff

$$\langle \mathrm{Im}\Pi_x^\sharp, E_x \rangle = T_x M, \quad x \in M. \tag{6}$$

Example

A locally conformal symplectic structure on an even-dimensional smooth manifold M consists in a pair (Ω, α) with Ω non-degenerate, α closed and

$$\mathrm{d}\Omega + \alpha \wedge \Omega = 0.$$

This results in a Jacobi pair (Π, E) with

$$\langle \rho \wedge \lambda, \Pi \rangle \equiv \langle \Omega, \Omega^\sharp \rho \wedge \Omega^\sharp \lambda \rangle, \quad E \equiv \Omega^\sharp \alpha.$$

By $\Omega^\sharp$ we denoted the inverse of the isomorphism

$$\Omega^\flat : \mathfrak{X}^1(M) \to \Omega^1(M), \quad \Omega^\flat X \equiv -i_X \Omega.$$

From Jacobi pairs to twisted Jacobi pairs

Example

A coorientable contact structure on an odd-dimensional smooth manifold M is given by a 1-form θ such that

$$\mu \equiv \theta \wedge (\mathrm{d}\theta)^m$$

is a volume form, i.e.,

$$\mu^\flat : \mathfrak{X}^1(M) \to \Omega^{2m}(M), \quad \mu^\flat X \equiv -i_X \mu$$

is an isomorphism. The pair (Π, E) is a Jacobi one where E is the Reeb vector field, i.e., the unique solution to

$$i_E\theta = 1, \quad i_E \mathrm{d}\theta = 0$$

and

$$\langle \mathrm{d}f \wedge \mathrm{d}g, \Pi \rangle \equiv \langle \mathrm{d}\theta, X_f \wedge X_g \rangle.$$

Example

Previously, by X_f we meant the Hamiltonian vector field associated with the smooth function $f \in \mathcal{F}(M)$ given by the considered coorientable contact structure, i.e., the unique solution to the equations

$$i_{X_f}\theta = f, \quad i_{X_f}\mathrm{d}\theta = i_E\left(\mathrm{d}f \wedge \theta\right).$$

From Jacobi pairs to twisted Jacobi pairs

Theorem

If a Jacobi pair (Π, E) *on a smooth manifold* M *is transitive then* M *is either a locally conformal symplectic manifold or a coorientable contact one.*

Theorem

The characteristic distribution of a Jacobi pair is completely integrable with the characteristic leaves either locally conformal symplectic manifolds or coorientable contact ones.

From Jacobi pairs to twisted Jacobi pairs

Let $(M, \mathcal{A}_M)$ be a smooth manifold. By definition, a twisted Jacobi pair $((\Pi, E), \omega)$ consists in

$$\Pi \in \mathfrak{X}^2(M), \quad E \in \mathfrak{X}^1(M), \quad \omega \in \Omega^2(M)$$

that enjoy the properties

$$\frac{1}{2}[\Pi, \Pi] + E \wedge \Pi = \Pi^\sharp \mathrm{d}\omega + \Pi^\sharp \omega \wedge E \tag{7}$$

$$[E, \Pi] = -\left(\Pi^\sharp i_E \mathrm{d}\omega + \Pi^\sharp i_E \omega \wedge E\right). \tag{8}$$

From Jacobi pairs to twisted Jacobi pairs

A twisted Jacobi pair endows the vector space $\mathcal{F}(M)$ with the $\mathbb{R}$-linear and skew-symmetric bracket

$$\{\bullet,\bullet\}:\mathcal{F}(M)\times\mathcal{F}(M)\to\mathcal{F}(M),$$
$$\{f,g\}\equiv i_{\Pi}\mathrm{d}f\wedge\mathrm{d}g+i_{E}(f\mathrm{d}g-g\mathrm{d}f), \tag{9}$$

which verifies

$$\{f,gh\}-g\{f,h\}-h\{f,g\}=gh\mathcal{L}_{E}f, \tag{10}$$

and

$$\mathrm{Jac}\{f,g,h\}=i_{\Pi^{\sharp}\mathrm{d}\omega+\Pi^{\sharp}\omega\wedge E}(\mathrm{d}f\wedge\mathrm{d}g\wedge\mathrm{d}h)$$
$$-i_{\Pi^{\sharp}i_{E}\mathrm{d}\omega+\Pi^{\sharp}i_{E}\omega\wedge E}(f\mathrm{d}g\wedge\mathrm{d}h+g\mathrm{d}h\wedge\mathrm{d}f+h\mathrm{d}f\wedge\mathrm{d}g), \tag{11}$$

From Jacobi pairs to twisted Jacobi pairs

The bracket in the above allows display the introduction of Hamiltonian vector fields

$$\mathcal{H} : \mathcal{F}(M) \to \mathfrak{X}^1(M),$$
$$\mathcal{H}(f) \equiv X_f = \Pi^\sharp \mathrm{d}f + fE, \tag{12}$$

which verify the relations

$$[X_f, X_g] - X_{\{f,g\}} = \Pi^\sharp i_{X_f \wedge X_g} \mathrm{d}\omega - (\mathcal{L}_E f)\, \Pi^\sharp i_{X_g}\omega + (\mathcal{L}_E g)\, \Pi^\sharp i_{X_f}\omega + \left(i_{X_f \wedge X_g}\omega\right) E. \tag{13}$$

$$[X_f, E] + X_{\mathcal{L}_E f} = \Pi^\sharp \left(i_{X_f \wedge E}\mathrm{d}\omega - (\mathcal{L}_E f)\, i_E\omega\right) + \left(i_{X_f \wedge E}\omega\right) E. \tag{14}$$

From Jacobi pairs to twisted Jacobi pairs

A twisted Jacobi pair $((\Pi, E), \omega)$ is said to be transitive if the characteristic distribution coincides with the tangent one i.e iff

$$\langle \mathrm{Im}\Pi^\sharp_x, E_x \rangle = T_x M, \quad x \in M. \tag{15}$$

Example

The pair (Ω, α), with Ω non-degenerate and α closed, is said to be a locally conformal symplectic structure twisted by $\omega \in \Omega^2(M)$ if

$$\mathrm{d}(\Omega - \omega) + \alpha \wedge (\Omega - \omega) = 0.$$

This results in a twisted Jacobi pair $((\Pi, E), \omega)$ with

$$\langle \rho \wedge \lambda, \Pi \rangle \equiv \langle \Omega, \Omega^\sharp \rho \wedge \Omega^\sharp \lambda \rangle, \quad E \equiv \Omega^\sharp \alpha.$$

By $\Omega^\sharp$ we denoted the inverse of the isomorphism

$$\Omega^\flat : \mathfrak{X}^1(M) \to \Omega^1(M), \quad \Omega^\flat X \equiv -i_X \Omega.$$

From Jacobi pairs to twisted Jacobi pairs

Example

The contact structure θ is said to be twisted by the 2-form ω if

$$\mu \equiv \theta \wedge (\mathrm{d}\theta + \omega)^m$$

is a volume form, i.e.,

$$\mu^\flat : \mathfrak{X}^1(M) \to \Omega^{2m}(M), \quad \mu^\flat X \equiv -i_X \mu$$

is an isomorphism. The structure $((\Pi, E), \omega)$ is a twisted Jacobi pair where E is the twisted Reeb vector field, i.e., the unique solution to

$$i_E \theta = 1, \quad i_E (\mathrm{d}\theta + \omega) = 0$$

and

$$\langle \mathrm{d}f \wedge \mathrm{d}g, \Pi \rangle \equiv \langle \mathrm{d}\theta, X_f \wedge X_g \rangle.$$

From Jacobi pairs to twisted Jacobi pairs

Example

Previously, by X_f we meant the twisted Hamiltonian vector field associated with the smooth function $f \in \mathcal{F}(M)$ given by the considered twisted coorientable contact structure, i.e., the unique solution to the equations

$$i_{X_f}\theta = f, \quad i_{X_f}(\mathrm{d}\theta + \omega) = i_E(\mathrm{d}f \wedge \theta).$$

From Jacobi pairs to twisted Jacobi pairs

Theorem

If a twisted Jacobi pair $((\Pi, E), \omega)$ *on a smooth manifold* M *is transitive then* M *is either a twisted locally conformal symplectic manifold or a twisted coorientable contact one.*

Theorem

The characteristic distribution of a twisted Jacobi pair is completely integrable with the characteristic leaves either twisted locally conformal symplectic manifolds or twisted coorientable contact ones.

Trivial line bundle formulation

- From Jacobi pairs to twisted Jacobi pairs
- Relaxing twisted Jacobi pairs: Jacobi pair with background
- Jacobi-like pairs as distinguished elements of a Lie algebroid

Relaxing twisted Jacobi pairs: Jacobi pair with background

Definition

A pair $((\Pi, E), (\phi, \omega))$ consisting in

$$\Pi \in \mathfrak{X}^2(M), \quad E \in \mathfrak{X}^1(M), \quad \phi \in \Omega^3(M), \quad \omega \in \Omega^2(M)$$

which enjoys the properties

$$\frac{1}{2}[\Pi, \Pi] + E \wedge \Pi = \Pi^\sharp \phi + \Pi^\sharp \omega \wedge E \tag{16}$$

$$[E, \Pi] = -\left(\Pi^\sharp i_E \phi + \Pi^\sharp i_E \omega \wedge E\right) \tag{17}$$

is called Jacobi pair (Π, E) with background (ϕ, ω).

It is immediate that if in the above we take

$$\phi \equiv \mathrm{d}\omega \tag{18}$$

then we recover the twisted Jacobi pair.

Relaxing twisted Jacobi pairs: Jacobi pair with background

Example

Let's consider the four-dimensional smooth manifold $\mathbb{R}^4$ with the global coordinates $x = \left(x^1, x^2, x^3, x^4\right)$ and the real smooth functions $f, e \in \mathcal{C}^\infty\left(\mathbb{R}^4\right)$ among which f is nowhere vanishing and

$$e = e\left(x^1, x^2\right).$$

The geometric objects

$$\Pi = \frac{1}{f}\left(\partial_1 \wedge \partial_4 + \partial_2 \wedge \partial_3\right),$$
$$E = -\frac{1}{f}\left(\left(\partial_1 e\right)\partial_4 + \left(\partial_2 e\right)\partial_3\right) = -\Pi^\sharp \mathrm{d}e, \quad \omega = 0,$$
$$\phi = \mathrm{d}\left(f\ \mathrm{d}x^2 \wedge \mathrm{d}x^3 + f\ \mathrm{d}x^1 \wedge \mathrm{d}x^4\right) - f\mathrm{d}\left(e\ \mathrm{d}x^2 \wedge \mathrm{d}x^3 + e\ \mathrm{d}x^1 \wedge \mathrm{d}x^4\right),$$

organize $\mathbb{R}^4$ as a Jacobi pair with background whose 3-form is non-closed and twisting 2-form ω vanishes.

Relaxing twisted Jacobi pairs: Jacobi pair with background

Example

Let's consider the same four-dimensional smooth manifold $\mathbb{R}^4$ and take the smooth functions a, b with a nowhere vanishing. We introduce the objects

$$\Omega = a\left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4\right) \quad \omega = a\ \mathrm{d}x^1 \wedge \mathrm{d}x^2,$$
$$\phi = \mathrm{d}\omega + (\mathrm{d}a + a\ \mathrm{d}b) \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4,$$
$$\Pi = -\frac{1}{a}\left(\partial_1 \wedge \partial_2 + \partial_3 \wedge \partial_4\right), \quad E = \Omega^\sharp \mathrm{d}b.$$

With these tools at hand $((\Pi, E), (\phi, \omega))$ is nothing but a Jacobi pair with background defined by 3-form ϕ and non-trivial twisting 2-form ω. The background 3-form is closed if and only if

$$(\mathrm{d}a + a\mathrm{d}b) \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 = 0.$$

Relaxing twisted Jacobi pairs: Jacobi pair with background

A Jacobi pair with background endows the vector space $\mathcal{F}(M)$ with the $\mathbb{R}$-linear and skew-symmetric bracket

$$\begin{aligned} &\{\bullet,\bullet\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M), \\ &\{f,g\} \equiv i_{\Pi}\mathrm{d}f \wedge \mathrm{d}g + i_E\left(f\mathrm{d}g - g\mathrm{d}f\right), \end{aligned} \tag{19}$$

which verifies

$$\{f,gh\} - g\{f,h\} - h\{f,g\} = gh\mathcal{L}_E f, \tag{20}$$

and

$$\begin{aligned} \mathrm{Jac}\{f,g,h\} &= i_{\Pi^\sharp\phi + \Pi^\sharp\omega\wedge E}\left(\mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h\right) \\ &\quad - i_{\Pi^\sharp i_E\phi + \Pi^\sharp i_E\omega\wedge E}\left(f\mathrm{d}g \wedge \mathrm{d}h + g\mathrm{d}h \wedge \mathrm{d}f + h\mathrm{d}f \wedge \mathrm{d}g\right) \end{aligned} \tag{21}$$

Relaxing twisted Jacobi pairs: Jacobi pair with background

The bracket in the above allows display the introduction of Hamiltonian vector fields

$$\begin{aligned} &\mathcal{H} : \mathcal{F}(M) \to \mathfrak{X}^1(M), \\ &\mathcal{H}(f) \equiv X_f = \Pi^\sharp \mathrm{d}f + fE, \end{aligned} \tag{22}$$

which verify the relations

$$\begin{aligned} [X_f, X_g] - X_{\{f,g\}} &= \Pi^\sharp i_{X_f \wedge X_g}\phi - (\mathcal{L}_E f)\,\Pi^\sharp i_{X_g}\omega \\ &\quad + (\mathcal{L}_E g)\,\Pi^\sharp i_{X_f}\omega + \left(i_{X_f \wedge X_g}\omega\right)E. \end{aligned} \tag{23}$$

$$[X_f, E] + X_{\mathcal{L}_E f} = \Pi^\sharp\left(i_{X_f \wedge E}\phi - (\mathcal{L}_E f)\, i_E\omega\right) + \left(i_{X_f \wedge E}\omega\right)E. \tag{24}$$

Relaxing twisted Jacobi pairs: Jacobi pair with background

A Jacobi pair with background $((\Pi, E), (\phi, \omega))$ is said to be transitive if its characteristic distribution coincides with the tangent one i.e iff

$$\langle \mathrm{Im}\Pi^\sharp_x, E_x \rangle = T_x M, \quad x \in M. \tag{25}$$

Example

A locally conformal symplectic structure (Ω, α), with Ω non-degenerate and α closed, is said to be with background (ϕ, ω) if

$$\phi = \mathrm{d}\Omega + \alpha \wedge (\Omega - \omega).$$

It generates a transitive Jacobi pair with background $((\Pi, E), (\phi, \omega))$ where

$$\langle \rho \wedge \lambda, \Pi \rangle = \left\langle \Omega, \Omega^\sharp \rho \wedge \Omega^\sharp \lambda \right\rangle, \quad E = \Omega^\sharp \alpha.$$

Relaxing twisted Jacobi pairs: Jacobi pair with background

Theorem

Let M be a smooth manifold and $((\Pi, E), (\phi_1, \omega_1))$ and $((\Pi, E), (\phi_2, \omega_2))$ be two Jacobi pairs with background on M. If both structures are transitive then the following alternative cases hold:

1. $\dim M$ *is even: there exists a 2-form,* $\omega \in \Omega^2(M)$, *such that*

$$\omega_1 = \omega_2 + \omega, \quad \phi_1 = \phi_2 - \omega \wedge \Pi^\flat E; \tag{26}$$

2. $\dim M$ *is odd:*

$$\omega_1 = \omega_2, \quad \phi_1 = \phi_2. \tag{27}$$

Relaxing twisted Jacobi pairs: Jacobi pair with background

Theorem

If a Jacobi pair with background $((\Pi, E), (\phi, \omega))$ on a smooth manifold M is transitive then M is either a locally conformal symplectic manifold with background or a twisted coorientable contact one.

Theorem

The characteristic distribution of a Jacobi pair with background is completely integrable with the characteristic leaves either locally conformal symplectic manifolds with background or twisted coorientable contact ones.

Relaxing twisted Jacobi pairs: Jacobi pair with background

The ‘Poissonization’ procedure also works for Jacobi pairs with background.

Definition

Let smooth manifold $(M, \mathcal{A}_M)$ endowed with a pair (Π, ϕ) consisting in

$$\Pi \in \mathfrak{X}^2(M), \quad \phi \in \Omega^3(M),$$

which verify

$$[\![\Pi, \Pi]\!] = 2\Pi^\sharp \phi \tag{28}$$

is called a Poisson manifold with background. If in addition there exists the vector field Z such that

$$\mathcal{L}_Z \Pi \equiv [\![Z, \Pi]\!] = -\Pi, \quad \mathcal{L}_Z \phi = \phi \tag{29}$$

then the Poisson manifold with background is said to be homogeneous.

Relaxing twisted Jacobi pairs: Jacobi pair with background

Theorem

If $((\Pi, E,), (\phi, \omega))$ *is a Jacobi pair with background, then the manifold*

$$\tilde{M} = M \times \mathbb{R} \tag{30}$$

can be naturally organized as a 'homogeneous' Poisson manifold with background defined by

$$\tilde{\Pi} = \mathrm{e}^{-\tau} (\Pi + \partial_\tau \wedge E), \quad \tilde{\phi} = \mathrm{e}^{\tau} (\phi + \omega \wedge \mathrm{d}\tau), \quad \tilde{Z} = \partial_\tau. \tag{31}$$

Trivial line bundle formulation

- From Jacobi pairs to twisted Jacobi pairs
- Relaxing twisted Jacobi pairs: Jacobi pair with background
- Jacobi-like pairs as distinguished elements of a Lie algebroid

Jacobi-like pairs as distinguished elements of a Lie algebroid

We start from the Lie algebroid

$$(TM \times \mathbb{R}, [\![\bullet,\bullet]\!], \rho)$$

with

$$[\![(X,f),(Y,g)]\!] \equiv ([X,Y], Xg - Yf), \quad \rho(X,f) \equiv X.$$

By means of the isomorphisms

$$\Gamma\left(\Lambda^{r+1}(TM \times \mathbb{R})\right) \simeq \mathfrak{X}^{r+1}(M) \times \mathfrak{X}^{r}(M),$$

its Gerstenhaber algebra $(\Gamma(\wedge^{\bullet}(TM \times \mathbb{R}), [\![\bullet,\bullet]\!]))$ reads

$$[\![(P,Q),(R,S)]\!] = ([P,R], [P,S] + (-)^{r}[Q,R]).$$

Jacobi-like pairs as distinguished elements of a Lie algebroid

Moreover, the differential of its de Rham complex $(\Gamma(\wedge^{\bullet}(TM \times \mathbb{R})^*), \mathbf{d})$, $\mathbf{d}$ can be written, by means of the isomorphisms

$$\Gamma\left(\Lambda^{r+1}(TM \times \mathbb{R})^*\right) \simeq \Omega^{r+1}(M) \times \Omega^{r}(M),$$

in terms of the standard de Rham differential, d as

$$\mathbf{d}(\omega, \alpha) \equiv (\mathrm{d}\omega, -\mathrm{d}\alpha), \quad (\omega, \alpha) \in \Omega^{k}(M) \times \Omega^{k-1}(M).$$

We consider the $\mathbf{d}$-cocycle $(0,1) \in \Omega^1(M) \times \mathcal{F}(M)$,

$$\mathbf{d}(0,1) = 0$$

and construct the Lie algebroid with 1-cocycle

$$\left(TM \times \mathbb{R}, [\![\bullet, \bullet]\!]^{(0,1)}, \rho\right).$$

Jacobi-like pairs as distinguished elements of a Lie algebroid

The previous bracket has the concrete expression

$$[\![(P,Q),(R,S)]\!]^{(0,1)} = (I, II)\,,$$

where

$$I \equiv [P,R] + p\,(-)^r\,P \wedge S - rQ \wedge R,$$

$$II \equiv [P,S] + (-)^r\,[Q,R] + (p-r)\,Q \wedge S.$$

Jacobi-like pairs as distinguished elements of a Lie algebroid

Within the previous context, the equations governing the Jacobi pair (Π, E) on the smooth manifold M simply read

$$[\![(\Pi, E), (\Pi, E)]\!]^{(0,1)} = 0.$$

Also, the equations exhibiting the twisted Jacobi pair $((\Pi, E), \omega)$ reduce to

$$[\![(\Pi, E), (\Pi, E)]\!]^{(0,1)} = 2\,(\Pi, E)^{\sharp}\,(\mathrm{d}\omega, \omega).$$

Finally, the equations displaying the Jacobi pair with background $((\Pi, E), (\phi, \omega))$ are

$$[\![(\Pi, E), (\Pi, E)]\!]^{(0,1)} = 2\,(\Pi, E)^{\sharp}\,(\phi, \omega).$$

Jacobi-like pairs as distinguished elements of a Lie algebroid

Previously, we denoted by $(\Pi, E)^{\sharp}$ the $\mathcal{F}(M)$-module morphism

$$(\Pi, E)^{\sharp} : \Gamma(\Lambda^{k}(T^{*}M \times \mathbb{R})) \to \Gamma(\Lambda^{k}(TM \times \mathbb{R})), \tag{32}$$

which is the linear extension of

$$\Omega^{1}(M) \times \mathcal{F}(M) \ni (\beta, f) \to (\Pi^{\sharp}\beta + fE, -i_{E}\beta) \in \mathfrak{X}^{1}(M) \times \mathcal{F}(M)$$

.

Outline

Non-trivial line bundle formulation

- Jacobi-like line bundles
- Jacobi-like line bundles encompass Jacobi-like pairs

Non-trivial line bundle formulation

- Jacobi-like line bundles
- Jacobi-like line bundles encompass Jacobi-like pairs

Jacobi-like line bundles

By definition, a Jacobi bundle consists in a line bundle $L \to M$ endowed with a bracket

$$\{\bullet, \bullet\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L),$$

that enjoys the properties:

- It is $\mathbb{R}$-linear and skew-symmetric;
- It verifies the Jacobi identity i.e.

$$\{s_1, \{s_2, s_3\}\} + \text{circular} = 0, \quad s_1, s_2, s_3 \in \Gamma(L) \tag{33}$$

- It is local i.e.

$$\text{supp}\{s_1, s_2\} \subset \text{supp} s_1 \cap \text{supp} s_2, \quad s_1, s_2 \in \Gamma(L) \tag{34}$$

Jacobi-like line bundles

It is immediate that Jacobi pairs are equivalent to trivial Jacobi bundles. Indeed, any Jacobi pair (Π, E) on a given manifold endows the trivial line bundle

$$\mathbb{R}_M \equiv \mathbb{R} \times M$$

$\mathcal{F}(M)$-module of smooth sections

$$\Gamma\left(\mathbb{R}_M\right) = \mathcal{F}\left(M\right)$$

with the bracket

$$\{f, g\} \equiv i_\Pi \mathrm{d}f \wedge \mathrm{d}g + i_E\left(f\mathrm{d}g - g\mathrm{d}f\right) \tag{35}$$

and conversely, any Jacobi structure on the trivial line bundle displays a bracket in $\mathcal{F}\left(M\right)$.

Jacobi-like line bundles

We consider the Lie algebroid $(DL, [\bullet, \bullet], \sigma)$ whose sections

$$\mathcal{D}(L) \equiv \Gamma(DL)$$

are nothing but the derivations of the module [over $\mathcal{F}(M)$] $\Gamma(L)$ i.e. $\mathbb{R}$-linear maps $\triangle$ which enjoy the existence of a [unique] vector field $X_{\triangle}$ such that

$$\triangle(fs) = (X_{\triangle} f)\, s + f \triangle s, \quad s \in \Gamma(L), f \in \mathcal{F}(M). \tag{36}$$

Previously, the bracket is given by

$$[\triangle, \triangle'] \equiv \triangle\triangle' - \triangle'\triangle,$$

while the anchor returns symbols of derivations

$$\sigma(\triangle) \equiv X_{\triangle}.$$

Jacobi-like line bundles

Now, associated with the tautological representation of DL on L

$$\nabla : \Gamma(DL) \longrightarrow \Gamma(DL), \quad \nabla_{\Box}\lambda \equiv \Box\lambda, \quad \lambda \in \Gamma(L), \tag{37}$$

the Jacobi algebroid (DL, L) is at hand. This is equivalent to

- the Gerstenhaber-Jacobi algebra consisting in the module

$$\mathcal{D}^{\bullet}L \equiv \Gamma(\wedge^{\bullet}J_1L \otimes L) \quad \text{over the algebra} \quad \Gamma(\wedge^{\bullet}J_1L) \tag{38}$$

- der-complex consisting in the module

$$\Omega_L^{\bullet} \equiv \Gamma(\wedge^{\bullet}(DL)^* \otimes L) \quad \text{over the algebra} \quad \Gamma(\wedge^{\bullet}(DL)^*) \tag{39}$$

In the above we used the notation

$$J_1L \equiv \left(J^1L\right)^* \tag{40}$$

and also the vector bundle isomorphism

$$J_1L \simeq DL \otimes L^*. \tag{41}$$

Jacobi-like line bundles

The homogeneous elements in the algebra (38) consists in skew-symmetric, first-order differential operators

$$\triangle : \Gamma(L) \times \cdots \times \Gamma(L) \to \mathcal{F}(M) \equiv \Gamma(\mathbb{R}_M) \tag{42}$$

while those of the module (38) are the skew-symmetric, first-order differential operators

$$\Box : \Gamma(L) \times \cdots \times \Gamma(L) \to \Gamma(L) \tag{43}$$

The bracket in the previous Gerstenhaber-Jacobi algebra reads

$$[\![\Box_1, \Box_2]\!] \equiv (-)^{k_1 k_2} \Box_1 \circ \Box_2 - \Box_2 \circ \Box_1, \quad \Box_a \in \mathcal{D}^{k_a+1} L, \tag{44}$$

with $\circ$ the Gerstenhaber multiplication

$$\Box_1 \circ \Box_2 (s_1, \cdots, s_{k_1+k_2+1}) \equiv$$
$$\sum_{\tau \in S_{k_1+1,k_2}} (-)^{\tau} \Box_1 \left(\Box_2 \left(s_{\tau(1)}, \cdots, s_{\tau(k_1+1)}\right), s_{\tau(k_1+2)}, \cdots, s_{\tau(k_1+k_2+1)}\right)$$

Jacobi-like line bundles

Concerning der-complex, $\Omega_L^\bullet$, it is endowed with a homological derivation, d_D, which symbol is nothing but de Rham differential, d_{DL}, associated with the Lie algebroid DL

$$\langle \mathrm{d}_D \lambda, \Box \rangle = \langle \Box, j^1 \lambda \rangle \tag{45}$$

$$\mathrm{d}_D (\omega \wedge \Omega) = (\mathrm{d}_{DL} \omega) \wedge \Omega + (-)^{|\omega|} \omega \wedge \mathrm{d}_D \Omega, \omega \in \Gamma \left(\wedge^\bullet (DL)^* \right), \Omega \in \Omega_L^\bullet \tag{46}$$

It can be shown that the cohomology of d_D in the der-complex is always trivial i.e.

$$\mathrm{d}_D \Omega_{k>0} = 0 \Longleftrightarrow \Omega_{k>0} = \mathrm{d}_D \Theta_{k-1}. \tag{47}$$

Jacobi-like line bundles

In this unified context, a Jacobi bundle consists in a line bundle $L \to M$ endowed with a first-order bi-differential operator

$$J \in \mathcal{D}^2 L$$

that verifies Maurer-Cartan equation

$$[\![J, J]\!] \equiv -2J \circ J = 0. \tag{48}$$

The connection between the bracket and the bi-differential operator J simply reads

$$\{s_1, s_2\} \equiv J\left(s_1, s_2\right), \quad s_1, s_2 \in \Gamma\left(L\right). \tag{49}$$

Jacobi-like line bundles

By means of the vector bundle morphism

$$\hat{J} : J^1 L \wedge J^1 L \to L, \quad \langle \hat{J}, j^1\lambda \wedge j^1\rho \rangle \equiv J(\lambda, \rho), \tag{50}$$

the Jacobi bundle $(L \to M, J)$ is said to be transitive if

$$\mathrm{Im}\left(\sigma \circ \hat{J}^\sharp\right) = TM.$$

Example

Let $\mathcal{K}$ be a contact structure on M ,i.e.,

$$\omega_\mathcal{K} : \mathcal{K} \times \mathcal{K} \to TM/\mathcal{K}, \quad \langle \omega_\mathcal{K}, X \wedge Y \rangle \equiv [X, Y] \mod \mathcal{K}$$

is non-degenerate. It defines a unique Jacobi bundle $(TM/\mathcal{K} \to M, J_\mathcal{K})$ which is transitive.

Jacobi-like line bundles

Example

An lcs structure on a given line bundle $L \to M$ is a pair (∇, Ω) consisting in a representation ∇ of the tangent Lie algebroid $TM \to M$ on a line bundle and a non-degenerate L-valued 2-form $\Omega \in \Omega^2(M; L)$ which is closed with respect to the homological degree 1 derivation d_∇ associated with the Jacobi algebroid structure $([\bullet, \bullet], \nabla)$ on the pair (TM, L),

$$d_\nabla \Omega = 0.$$

It defines a unique transitive Jacobi bundle $(L \to M, J)$ with

$$J(\lambda, \mu) \equiv \langle \Omega, \Omega^\sharp(\mathrm{d}_\nabla \mu) \wedge \Omega^\sharp(\mathrm{d}_\nabla \lambda) \rangle.$$

Jacobi-like line bundles

Moreover, a twisted Jacobi bundle consists in a line bundle $L \to M$ endowed with a first-order bi-differential operator

$$J \in \mathcal{D}^2 L$$

which 'nilpotency' (48) is 'twisted' via the closed Atiyah 3-form

$$\Phi \in \Omega_L^3, \quad \mathrm{d}_D \Phi = 0, \tag{51}$$

i.e.

$$[\![J, J]\!] = 2\hat{J}^\sharp \Phi. \tag{52}$$

Also here, the twisted Jacobi bundle $(L \to M, J, \Phi)$ is said to be transitive if

$$\mathrm{Im}\left(\sigma \circ \hat{J}^\sharp\right) = TM.$$

Jacobi-like line bundles

Example

A hyperplane distribution $\mathcal{K}$ together with a 2-form $\psi \in \Gamma\left(\wedge^2\mathcal{K}^* \otimes L\right)$, $L \equiv TM/\mathcal{K}$ is said to be a twisted contact structure on M if

$$\omega_{\mathcal{K}} + \psi \in \Gamma\left(\wedge^2\mathcal{K}^* \otimes L\right)$$

is non-degenerate. It defines a unique twisted Jacobi bundle $(L \to M, J_{\mathcal{K},\psi}, \Omega_{\mathcal{K},\psi})$ which is transitive.

Example

A twisted lcs structure on a given line bundle $L \to M$ is pair $((\nabla, \Omega), \omega)$ consisting in a representation ∇ of the tangent Lie algebroid $TM \to M$ on a line bundle, a non-degenerate L-valued 2-form $\Omega \in \Omega^2(M; L)$ and an L-valued 2-form $\omega \in \Omega^2(M; L)$ which verify the compatibility condition

$$d_\nabla \Omega = d_\nabla \omega.$$

It defines a unique transitive twisted Jacobi bundle $(L \to M, J, \mathrm{d}_D \sigma^* \omega)$ with

$$J(\lambda, \mu) \equiv \langle \Omega, \Omega^\sharp(\mathrm{d}_\nabla \mu) \wedge \Omega^\sharp(\mathrm{d}_\nabla \lambda) \rangle.$$

Jacobi-like line bundles

Finally, a Jacobi bundle with background consists in a line bundle $L \to M$ endowed with a first-order bi-differential operator

$$J \in \mathcal{D}^2 L$$

which 'nilpotency' (48) is 'broken' via an Atiyah 3-form

$$\Phi \in \Omega^3_L, \tag{53}$$

i.e.

$$[\![J, J]\!] = 2 \hat{J}^\sharp \Phi. \tag{54}$$

Also here, Jacobi bundle with background $(L \to M, J, \Phi)$ is said to be transitive if

$$\mathrm{Im}\left(\sigma \circ \hat{J}^\sharp\right) = TM.$$

Jacobi-like line bundles

Example

An lcs structure with background on a given line bundle $L \to M$ is pair $((\nabla, \Omega), (\phi, \omega))$ consisting in a representation ∇ of the tangent Lie algebroid $TM \to M$ on a line bundle, a non-degenerate L-valued 2-form $\Omega \in \Omega^2(M; L)$ an L-valued 3-form $\phi \in \Omega^3(M; L)$ and an L-valued 2-form which verify the compatibility condition

$$d_{\nabla}\Omega = d_{\nabla}\omega + \phi.$$

It defines a unique transitive Jacobi bundle with background$(L \to M, J, \mathrm{d}_D \sigma^* \omega + \sigma^* \phi)$ with

$$J(\lambda, \mu) \equiv \langle \Omega, \Omega^\sharp(\mathrm{d}_\nabla \mu) \wedge \Omega^\sharp(\mathrm{d}_\nabla \lambda) \rangle.$$

Non-trivial line bundle formulatioe

- Jacobi-like line bundles
- Jacobi-like line bundles encompass Jacobi-like pairs

Jacobi-like line bundles encompass Jacobi-like pairs

When the line bundle $L \to M$ is trivial

$$L \equiv \mathbb{R}_M,$$

by means of the isomorphism

$$D\mathbb{R}_M = TM \times \mathbb{R}, \tag{55}$$

the homogeneous elements of the Gerstenhaber-Jacobi algebra (38) reduce to

$$\mathcal{D}^0\mathbb{R}_M = \mathcal{F}(M) \quad \mathcal{D}^k\mathbb{R}_M = \mathfrak{X}^k(M) \times \mathfrak{X}^{k-1}(M), \quad k > 0 \tag{56}$$

while the Gerstenhaber-Jacobi bracket becomes

$$[\![\bullet, \bullet]\!]^{(0,1)}.$$

Jacobi-like line bundles encompass Jacobi-like pairs

In addition, the homogeneous elements of the (Atiyah)der-complex read

$$\Omega^0_{\mathbb{R}_M} = \mathcal{F}(M) \quad \Omega^k_{\mathbb{R}_M} = \Omega^k(M) \times \Omega^{k-1}(M), \quad k > 0. \tag{57}$$

Moreover, the homological derivation in the Atiyah complex can be written in terms of de Rham differential like

$$\mathrm{d}_D f \equiv \mathrm{d}f, \quad \mathrm{d}_D(\omega_k, \omega_{k-1}) \equiv (\mathrm{d}\omega_k, \omega_k - \mathrm{d}\omega_{k-1}), \quad k > 0. \tag{58}$$

With these identifications at hand, the bi-differential operator J is realised as

$$J \leftrightarrow (\Pi, E) \in \mathfrak{X}^2(M) \times \mathfrak{X}^1(M),$$

the Atiyah 3-form in the twisted Jacobi bundle (51) becomes

$$\Phi \leftrightarrow (\mathrm{d}\omega, \omega) = \mathrm{d}_D(\omega, 0) \in \Omega^3(M) \times \Omega^2(M),$$

while the Atiyah 3-form in the Jacobi bundle with background reads

$$\Phi \leftrightarrow (\phi, \omega) \in \Omega^3(M) \times \Omega^2(M).$$

References

- E.M. Cioroianu, C. Vizman, A linear algebraic setting for Jacobi structures, Journal of Geometry and Physics **159** (2021) 103904
- E.M. Cioroianu, L. Vitagliano, A.G. Tortorella, C. Vizman, The linear algebra of (Dirac-) Jacobi geometry, The West University of Timișoara Publishing House, Timișoara, 2021, ISBN: 978-973-125-821-8
- E.M. Cioroianu, C. Vizman, Jacobi structures with background, *Int. J. Geom. Meth. Mod. Phys.* **17** (2020) 2050063
- E.M. Cioroianu, Contact Structures: From Standard to Line Bundle Approach, AIP Conference Proceedings 2071 (2019) 020006
- E.M. Cioroianu, Locally Conformal Symplectic Structures: From Standard to Line Bundle Approach, AIP Conference Proceedings 2218 (2020) 050007
- E.M. Cioroianu, Twisted Jacobi versus Jacobi with background structures, *Revue Roumaine de Mathématiques Pures et Appliquées* **66(2)** (2021) 457–484
- E.M. Cioroianu, Jacobi bundles with background, *work in progress*