Entropy of Reissner-Nordström-like black holes

Milutin Blagojević and Branislav Cvetković[1]

[1] Institute of physics, University of Belgrade

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Talk outline

Talk is based on the paper:

- M. Blagojević, B. Cvetković, under review

- The discovery of the thermodynamic behavior of black holes in general relativity (GR) has deeply affected our understanding of the gravitational dynamics.
- At the core of that behavior lies the concept of entropy, which is classically represented by a boundary term Γ_H, interpreted as the Noether charge on black hole horizon.
- For a class of spherically symmetric black holes in GR without matter, $\delta\Gamma_H$ has the standard form $T\delta S$, where T is the black hole temperature and S entropy, equal to one-fourth of the horizon area.
- This result remains essentially valid for diffeomorphism invariant Lagrangian theories of gravity, based on Riemannian geometry of spacetimes. Such a universal nature of S is kind of a puzzle, as one might naturally expect that entropy would depend on the structure of black holes and the type of the underlying gravitational dynamics.

- Entropy and the asymptotic charges are strongly interrelated through the first law of black hole thermodynamics.
- A Hamiltonian approach to entropy was proposed in:
  - M. Blagojević and B. Cvetković, Phys. Rev. D 99, 104058 (2019),

  in which the asymptotic charges and entropy are described in a unified manner as the canonical charges at infinity and horizon, respectively.
- This approach was primarily intended to describe entropy in the framework of Poincaré gauge theory (PG), where *both the curvature and the torsion* are essential ingredients of the gravitational dynamics.
- However, for a number of black holes in PG, including Kerr-AdS back holes *with or without torsion*, it was found, somewhat unexpectedly, that entropy retains its area form.

- Are there black holes for which entropy and/or the first law change their standard forms? There are two 3-dimensional PG models which are interesting in this respect. First, entropy of the Banados-Teitelboim-Zanelli black hole with torsion is found to depend on the parameter that measures the strength of torsion , and second, entropy of the Oliva-Tempo-Troncoso black hole depends on a “hair" parameter appearing in the metric . In both cases, the first law remains unchanged.
- This situations motivates us to explore thermodynamic aspects of a new black hole with torsion, found in the four-dimensional PG by Cembranos and Valcarcel
- It represents a *gravitational analogue* of the standard Reissner-Nordström black hole, in which the “electric charge" is produced not by a source, but by the *gravitational field in vacuum.*

Notations and conventions

- The Latin indices $(i, j, \dots)$ are the local Lorentz indices, the Greek indices $(\mu, \nu, \dots)$ are the coordinate indices, and both run over $0, 1, 2, 3$.
- The orthonormal coframe (tetrad) 1-form is ϑ^i, and ω^{ij} is a metric compatible (Lorentz) connection 1-form.
- The metric components in the local Lorentz basis are $\eta_{ij} = (1, -1, -1, -1)$.
- The Hodge dual of a form α is denoted by ${}^\star\alpha$, and the wedge product of forms is implicitly understood.

- Poincaré gauge theory is a gauge theory of gravity, in which the tetrad field ϑ^i and the Lorentz connection ω^{ij} are found to be the gauge potentials.
- The related field strengths are naturally identified with the torsion $T^i = d\vartheta^i + \omega^i{}_k \vartheta^k$ and the curvature $R^{ij} = d\omega^{ij} + \omega^i{}_k \omega^{kj}$ of the Riemann-Cartan (RC) spacetime.
- In vacuum the PG dynamics is completely determined by the gravitational Lagrangian, which is assumed to be parity invariant and at most quadratic in the field strengths,

$$L_G = -{}^\star(a_0 R + 2\Lambda) + T^i \sum_{n=1}^{3} {}^\star(a_n\, {}^{(n)}T_i) + \frac{1}{2} R^{ij} \sum_{n=1}^{6} {}^\star(b_n\, {}^{(n)}R_{ij}) \,. \tag{2.1}$$

Here, ${}^{(n)}T^i$ and ${}^{(n)}R^{ij}$ are irreducible parts of the field strengths, and (Λ, a_0, a_n, b_n) are the gravitational coupling constants.

- The gravitational field equations, obtained by varying L_G with respect to ϑ^i and ω^{ij}, can be written in a compact form

$$\delta\vartheta^i: \qquad \nabla H_i + E_i = 0\,, \tag{2.2a}$$

$$\delta\omega^{ij}: \qquad \nabla H_{ij} + E_{ij} = 0\,, \tag{2.2b}$$

where $H_i := \partial L_G/\partial T^i$ and $H_{ij} := \partial L_G/\partial R^{ij}$ are the covariant momenta (2-forms), and $E_i := \partial L_g/\partial\vartheta^i$ and $E_{ij} := \partial L_G/\partial\omega^{ij}$ are the energy-momentum and spin currents (3-forms), respectively.

- In the early 1970s, Regge and Teitelboim demonstrated that the canonical analysis of GR requires the standard ADM Hamiltonian to be modified by adding suitable *boundary terms* at infinity to ensure the consistency of the variational formalism. Their values are associated to the asymptotic conserved charges (energy-momentum or angular momentum).

- Further generalization of these ideas within PG was focussed on understanding entropy as an additional conserved charge at horizon.
- Let Σ be a three-dimensional subspace of a stationary black hole spacetime, such that its boundary has two components, one at infinity, S_∞, and one at horizon, S_H.
- Then, for any Killing vector ξ, the associated boundary terms Γ_∞ and Γ_H are determined by:

$$\delta\Gamma_\infty = \oint_{S_\infty} \delta B(\xi)\,, \qquad \delta\Gamma_H = \oint_{S_H} \delta B(\xi)\,, \tag{2.3}$$

$$\delta B(\xi) := (\xi\rfloor\vartheta^i)\delta H_i + \delta\vartheta^i(\xi\rfloor H_i) + \frac{1}{2}(\xi\rfloor\omega^{ij})\delta H_{ij} + \frac{1}{2}\delta\omega^{ij}(\xi\rfloor\delta H_{ij})\,.$$

- Γ_∞ is a direct extension of the Regge-Teitelboim construction to PG, whereas Γ_H defines an additional contribution associated to entropy.

- The variation δ is required to satisfy the following rules:
  - **(r1)** On S_∞, the variation δ acts on the parameters of a black hole solution, but not on the parameters of the background configuration.
  - **(r2)** On S_H, the variation δ must keep surface gravity constant.
- When the asymptotic conditions allow the solutions for Γ_∞ and Γ_H to exist and be finite, they define the asymptotic charges and black hole entropy, respectively.
- Technically, Eqs. (2.3) are derived by requiring that the canonical gauge generator G is a regular (differentiable) functional on the phase space with given asymptotic conditions. The regularity of G is ensured by the relation

$$\delta\Gamma \equiv \delta\Gamma_\infty - \delta\Gamma_H = 0\,, \tag{2.4}$$

  which is interpreted as the first law of black hole thermodynamics.

- Classical solutions of PG help us to better understand the influence of torsion on the gravitational dynamics.
- We shall focus our attention on the static and spherically symmetric solution found by Cembranos and Valcarcel, in which the standard electric charge term in the Reissner-Nordström metric is imitated by a torsion parameter.
- The metric of asymptotically flat Reissner-Norström-like black holes has the form

$$ds^2 = N^2 dt^2 - \frac{dr^2}{N^2} - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \quad N^2 = 1 - \frac{2m}{r} + \frac{q}{r^2}\,,$$

where m is the usual mass parameter of the Schwarzschild metric, but q differs from the standard e^2 term in the Reissner-Nordström metric, which will be expressed on shell in terms of the Lagrangian coupling constants and a particular torsion parameter.

- The outer horizon (provided $q \le m^2$) is located at the larger root of $N^2 = 0$. The surface gravity is given by

$$\kappa = \frac{1}{2}\partial_r N^2\Big|_{r_+} = \frac{mr_+ - q}{r_+^3} = \frac{r_+^2 - q}{2r_+^3}\,. \tag{3.1}$$

- The orthonormal tetrad is chosen in the form

$$\vartheta^0 = Ndt\,, \quad \vartheta^1 = \frac{dr}{N}\,, \quad \vartheta^2 = rd\theta\,, \quad \vartheta^3 = r\sin\theta d\varphi\,. \tag{3.2}$$

- The ansatz for torsion is assumed to be static and rotationally invariant. The RC connection ω^{ij} reads

$$\begin{aligned} &\omega^{01} = -N'\vartheta^1\,, \quad \omega^{0c} = -\omega^{1c} = -\frac{N}{2r}\vartheta^c\,, \qquad c = (2,3)\,, \\ &\omega^{23} = \frac{\cos\theta}{r\sin\theta}\vartheta^3 + \frac{p}{Nr}\vartheta^-\,. \end{aligned} \tag{3.3}$$

p is a new parameter appearing in the torsion function and $\vartheta^- = \vartheta^0 - \vartheta^1$.

- All irreducible parts of the torsion and curvature are nonvanishing. The torsion and the curvature define three topological invariants (Euler, Pontryagin and Nieh-Yan):

$$
I_E = \varepsilon_{ijmn} R^{mn} R^{ij} = 0\,, \qquad I_P = R^{ij} R_{ij} = -\frac{4p}{r^4}\,\hat{\epsilon}\,,
$$
$$
I_{NY} = T^i T_i - R_{ij}\vartheta^i \vartheta^j = \frac{2p}{r^2}\,\hat{\epsilon}\,, \tag{3.4}
$$

where $\hat{\epsilon} := \vartheta^0\vartheta^1\vartheta^2\vartheta^2$ is the volume 4-form.
- The solution is obtained by imposing the restrictions on the parameters in the Lagrangian, which takes the form:

$$
\begin{aligned}
L_G \;=\;& -{}^\star(a_0 R) + a_0 T^{i\star}\left({}^{(1)}T_1 - 2\,{}^{(2)}T_i - \frac{1}{2}\,{}^{(3)}T_i\right) \\
&+\frac{1}{2} b_2 R^{ij\star}\left({}^{(2)}R_{ij} - {}^{(3)}R_{ij} - \frac{1}{3}\,{}^{(5)}R_{ij}\right)\,,
\end{aligned} \tag{3.5}
$$

- The gravitational field equations determine the metric parameter q as

$$q = -\frac{b_2}{3a_0} p^2 \,. \tag{3.6}$$

- The quadratic torsion part of L_G is chosen to be the same as in the teleparallel equivalent of GR, and, as a consequence, the first two parts in L_G reduce to $-a_0\tilde{R}$ (Riemannian scalar curvature), up to a divergence.
- The metric parameter $q \sim p^2$ is produced by the p-dependent connection in the quadratic curvature part. The connection/torsion parameter p cannot be interpreted as the canonical charge, but it has an interesting relation to the topological invariants.

- Basic elements of the general variational equation for the asymptotic charges and entropy are a Killing vector $\xi = \partial_t$, the dynamical variables and the covariant momenta:

$$
\begin{aligned}
&H_0 = -H_1 = \frac{2a_0 p}{Nr}\vartheta^0\vartheta^1 - 2a_0\frac{N}{r}\vartheta^2\vartheta^3\,,\\
&H_2 = -\frac{a_0}{Nr^3}(q-r^2)\vartheta^-\vartheta^3\,, \quad H_3 = \frac{a_0}{Nr^3}(q-r^2)\vartheta^-\vartheta^2\,,\\
&H_{01} = -2a_0\vartheta^2\vartheta^3 - \frac{2b_2 p}{3r^2}\vartheta^0\vartheta^1\,, \quad H_{12} = -2a_0\vartheta^0\vartheta^3 + \frac{b_2 p}{3r^2}\vartheta^-\vartheta^2\,,\\
&H_{02} = 2a_0\vartheta^1\vartheta^3 + \frac{b_2 p}{3r^2}\vartheta^-\vartheta^2\,, \quad H_{13} = 2a_0\vartheta^0\vartheta^2 + \frac{b_2 p}{3r^2}\vartheta^-\vartheta^3\,,\\
&H_{03} = -2a_0\vartheta^1\vartheta^2 + \frac{b_2 p}{3r^2}\vartheta^-\vartheta^3\,, \quad H_{23} = -2a_0\vartheta^0\vartheta^1 - \frac{4b_2 p}{3r^2}\vartheta^2\vartheta^3\,.
\end{aligned}
$$

- The expression for energy E_t is determined from the relation $\delta E_t := \delta\Gamma_\infty$, where only the parameters of the solution, m and q (or equivalently p), are varied.

- By summing up the contributions (with $16\pi a_0 = 1$) we get:

$$\delta E_t \equiv \delta\Gamma_\infty = \delta m\,. \tag{4.1}$$

- Relying on the relations valid on horizon

$$N\partial_r N\big|_{r_+} = \kappa\,, \qquad N\delta N\big|_{r_+} = 0\,,$$

and for $\xi = \partial_t$, one finds the following:

$$\delta\Gamma_H = \frac{1}{2}\kappa\delta r_+^2 + \frac{1}{2r_+}\delta q\,. \tag{4.2}$$

- Usually, entropy is identified from the value of $\delta\Gamma_H$ by:

$$T\delta S = \delta\Gamma_H \quad \Leftrightarrow \quad \delta S = \pi\delta r_+^2 + \frac{\pi}{\kappa r_+}\delta q\,. \tag{4.3}$$

- However, this relation cannot be integrated to obtain S as a local function of the parameters m and q, or equivalently, of r_+ and r_-.

- The only consistent option is to identify entropy from the first, δ-integrable term of $\delta\Gamma_H$,

$$T\delta S = \frac{\kappa}{2}\pi\delta r_+^2\,, \qquad S := \pi r_+^2\,, \tag{4.4}$$

so that the complete $\delta\Gamma_H$ reads

$$\delta\Gamma_H = T\delta S + \frac{1}{2r_+}\delta q\,. \tag{4.5}$$

- Having found $\delta\Gamma_\infty$ and $\delta\Gamma_H$, we now wish to verify the validity of the first law,

$$\delta\Gamma_\infty = \delta\Gamma_H\,. \tag{4.6}$$

- A simple algebraic manipulation yields a relation (Smarr formula) between the thermodynamic variables (m, S, q):

$$m = \kappa r_+^2 + \frac{q}{r_+} \equiv 2TS + \frac{q}{r_+}\,. \tag{4.7}$$

- The variation of the of the Smarr formula yields the identity

$$\delta m = \frac{1}{2}\kappa\delta r_+^2 + \frac{1}{2r_+}\delta q\,, \tag{4.8}$$

which proves the validity of the first law.
- For $q > 0$, it is instructive to introduce the notation $q = c^2$ and rewrite (4.8) in a suggestive form

$$\delta m = T\delta S + \Phi\delta c\,, \qquad \Phi := \frac{c}{r_+}\,, \tag{4.9}$$

which formally coincides with the first law for the genuine Reissner-Nordsröm black hole in GR.
- However, the parameter c is not an ordinary electric charge or any other conserved charge, but just the *hair parameter* of the solution.
- Thus, the above form of the first law depends on the contribution of the hair variable c.

- We applied the Hamiltonian approach to analyze energy, entropy and the first law of the Reissner-Nordström-like black hole with torsion. The metric has the standard form, but the electric charge, stemming from the matter source, is replaced by a genuine gravitational parameter p.
- Entropy of the black hole is determined by the variation of the boundary term at horizon. S cannot be obtained from the usual formula $T\delta S = \delta\Gamma_H$, as it does not define S as a local function of the thermodynamic variables (m, q).
- The problem is resolved by focussing on the δ-integrable part of $\delta\Gamma_H$, which implies that entropy is one-fourth of the horizon area, $S = \pi r_+^2$.
- Thus, although the general formula for entropy is changed, its value retains the standard form. At the same time, the standard form of the first law is modified by an extra contribution stemming from the hair parameter $q \equiv c^2$.