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L_∞-Algebras and Braided Field Theory

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based on:

MDC, G. Giotopoulos, V. Radovanovic, R. J. Szabo, *L_∞-Algebras of Einstein-Cartan-Palatini Gravity*, JMP 61, 112502 (2020), arXiv: 2003.06173.

MDC, G. Giotopoulos, V. Radovanovic, R. J. Szabo, *Braided L_∞-Algebras, Braided Field Theory and Noncommutative Gravity*, arXiv:2103.08939.

MDC, N. Konjik, V. Radovanovic, R. J. Szabo, M. Toman, in preparation.

Overview

Motivation: general

Divergences in QFT, Early Universe, singularities of BHs $\Rightarrow$ QG $\Rightarrow$ Quantum space-time

One possibility: Noncommutative (NC) and/or nonassociative (NA) space-time.

There are different ways to work with NC geometry: spectral geometry of A. Connes, operators and representation theory (J. Madore, S. Majid...), defformation quantization (Kontsevich...) and the $\star$-product approach.

General Relativity (GR) is based on the diffeomorphism symmetry. The concept of space-time symmetry is difficult to generalize to NC/NA spaces. Different approaches realted to different ways of deforming classical space-time symmetries.

Drinfeld twist formalsim: a well defined way to deform a (Hopf) algebra of classical symmetries to a twisted (noncommutative, defomed) Hopf algebra, corresponding to a NC space-time.

Ambiguites in construction of NC actions (different orderings of noncommutative products fields). Using the (braided) L_∞ algebra in these constructions results in unique (sometimes unexpected) definitions of actions, EoM...

In this talk: star-product approach: represent abstract (noncommutative) algebra of coordinates on the commutative (classical) space-time, but keep the information about noncommutativity. NC multiplication in the algebra is maped to the $\star$-product

$$\begin{aligned} \hat{f}(\hat{x}) &\quad\mapsto\quad f(x) \\ \hat{f}(\hat{x})\hat{g}(\hat{x}) = \hat{f}\cdot\hat{g}(\hat{x}) &\quad\mapsto\quad f\star g(x). \end{aligned}$$

A well known example: θ-constant NC space: Moyal-Weyl $\star$-product

$$f \star g\,(x) = \sum_{n=0}^{\infty}\Big(\frac{i}{2}\Big)^n \frac{1}{n!}\theta^{\rho_1\sigma_1}\ldots\theta^{\rho_n\sigma_n}\Big(\partial_{\rho_1}\ldots\partial_{\rho_n}f(x)\Big)\Big(\partial_{\sigma_1}\ldots\partial_{\sigma_n}g(x)\Big)$$

$$f\cdot g + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho f)\cdot(\partial_\sigma g) + \mathcal{O}(\theta^2) \neq g\star f = \mathrm{R}_k g \star \mathrm{R}^k f.$$

$[x^\mu \overset{\star}{,} x^\nu] = i\theta^{\mu\nu}$, represents the NC algebra of coordinate operators.
Associative, noncommutative: $\mathcal{R}$-matrix $\mathcal{R} = \mathrm{R}^k \otimes \mathrm{R}_k$ and $\mathcal{R}^{-1} = \mathrm{R}_k \otimes \mathrm{R}^k$ encodes the noncommutativity of the $\star$-product.

$\star$-products can be obtained in different ways: deformational quantization [Bayen et al. '78], formality map [Kontsevich '98], twist formalism [Aschieri et al. '06, '08]. Differential geometry naturally deformed to NC differential geometry in the twist formalism.

NC geometry via the twist deformation

Start from a symmetry algebra g and its universal covering algebra Ug. Then define a twist operator $\mathcal{F}$ as:
-an invertible element of $Ug \otimes Ug$
-fulfills the 2-cocycle condition (ensures the associativity of the $\star$-product).

$$\mathcal{F} \otimes 1 (\Delta \otimes \mathrm{id})\mathcal{F} = 1 \otimes \mathcal{F}(\mathrm{id} \otimes \Delta)\mathcal{F}. \tag{1}$$

-additionaly: $\mathcal{F} = 1 \otimes 1 + \mathcal{O}(h)$; h-deformation parameter.

Braiding (noncommutativity): controled by the R-matrix $\mathcal{R} = \mathcal{F}^{-2} = \mathrm{R}^k \otimes \mathrm{R}_k$; triangular $\mathcal{R}_{21} = \mathcal{R}^{-1} = \mathrm{R}_k \otimes \mathrm{R}^k$.

In pactice: $U\mathrm{Vec}(\mathcal{M})$-module algebra $\mathcal{A}$ (functions, forms, tensors) and $a, b \in \mathcal{A}$, $\xi \in \mathrm{Vec}(\mathcal{M})$

$$\xi(ab) = \xi(a)b + a\xi(b), \qquad \text{Lie derivative, Leibinz rule (coproduct).}$$

The twist: $U\mathrm{Vec}(\mathcal{M}) \to U\mathrm{Vec}^{\mathcal{F}}(\mathcal{M})$ and $\mathcal{A} \to \mathcal{A}_\star$ with

$$a \cdot b \to a \star b = \cdot \circ \mathcal{F}^{-1}(a \otimes b) = \bar{\mathrm{f}}^{\alpha}(a) \cdot \bar{\mathrm{f}}_{\alpha}(b).$$

Commutativity: $a \star b = \bar{\mathrm{f}}^{\alpha}(a) \cdot \bar{\mathrm{f}}_{\alpha}(b) = \mathrm{R}_k(b) \star \mathrm{R}^k(a)$.

$\mathcal{A}_\star$ is a $U\mathrm{Vec}^{\mathcal{F}}(\mathcal{M})$-module algebra: $\tilde{\xi}(a \star b) = \tilde{\xi}_{(1)}(a) \star \tilde{\xi}_{(2)}(b)$, using the twisted coproduct $\Delta^{\mathcal{F}}\tilde{\xi} = \tilde{\xi}_{(1)} \otimes \tilde{\xi}_{(2)}$.

Motivation: a simple example of two commutators

Starting from a non-Abelian geuge theory with $\delta_\rho A = \mathrm{d}\rho + i[\rho, A]$, how can we define a NC gauge theory in the $\star$-product approach?

- $\star$-gauge theory

$$\delta_\rho A \to \delta^\star_\rho A = \mathrm{d}\rho + i[\rho \overset{\star}{,} A] = \mathrm{d}\rho + i(\rho \star A - A \star \rho),$$
$$\delta^\star_\rho(\phi_1 \star \phi_2) = \delta^\star_\rho \phi_1 \star \phi_2 + \phi_1 \star \delta^\star_\rho \phi_2.$$

  $[\,\overset{\star}{,}\,]$ no longer closes in the Lie algebra. It closes in the universal enveloping algebra (infinitely dimensional). Not compatible with the twisted diffeomorphism symmetry. These symmetries naturally appear in string theory.

- Braided gauge theory

$$\delta_\rho A \to \delta^\star_\rho A = \mathrm{d}\rho + i[\rho, A]_\star = \mathrm{d}\rho + i(\rho \star A - \mathrm{R}_k A \star \mathrm{R}^k \rho),$$
$$[\rho, A]_\star = \rho \star A - \mathrm{R}_k A \star \mathrm{R}^k \rho = \rho^a \star A^b [T^a, T^b] = i f^{abc} \rho^a \star A^b T^c,$$
$$\delta^\star_\rho(\phi_1 \star \phi_2) = \delta^\star_\rho \phi_1 \star \phi_2 + \mathrm{R}_k \phi_1 \star \delta^\star_{\mathrm{R}^k \rho} \phi_2.$$

  Braided gauge transformations close in the Lie algebra. They are compatible with the twisted diffeomorphisms. So far, the braided symmetries have not been seen in string theory.

How do we construct NC theory (actions, equations of motion...)? In general, there are ambiguities (usually in the interaction terms) due to the noncommutativity of the $\star$-product:

$$\bar{\psi}\gamma^{\mu}A_{\mu}\psi \rightarrow \bar{\psi}\star\gamma^{\mu}A_{\mu}\star\psi, \quad \bar{\psi}\gamma^{\mu}\star\psi\star A_{\mu}, \quad A_{\mu}\star\bar{\psi}\star\gamma^{\mu}\psi$$

Idea: use the well defined structure of a (braided) L_{∞}-algebra to formulate NC equations of motion and NC actions.

L_{∞}-algebra (strong homotopy algebra): generalization of a Lie algebra with higher order brackets.

-Higher spin gauge theories with field-dependent gauge parameters [Berends, Burgers, van Dam '85]

$$(\delta_{\alpha}\delta_{\beta}-\delta_{\beta}\delta_{\alpha})\Phi=\delta_{C(\alpha,\beta,\Phi)}\Phi.$$

-Generalized gauge symmetries of closed string field theory involve higher brackets [Zwiebach '15].

-Any classical field theory with generalized gauge symmetries is determined by an L_{∞}-algebra, due to duality with BV-BRST [Hohm, Zwiebach 17; Jurčo, Raspollini, Sämann, Wolf 18].

-L_{∞}-algebras of ECP gravity, classical and noncommutative [MDC, Giotopoulos, Radovanović, Szabo '20, '21].

L_∞ algebra and gauge field theory

L_∞-algebra: $\mathbb{Z}$-graded vector space $V = \bigoplus_{k\in\mathbb{Z}} V_k$ with graded antisymmetric multilinear maps, n-brackets

$$\ell_n : \bigotimes^n V \longrightarrow V \ , \quad v_1 \otimes \cdots \otimes v_n \longmapsto \ell_n(v_1, \ldots, v_n)$$

$$\ell_n(\ldots, v, v', \ldots) = -(-1)^{|v|\,|v'|}\, \ell_n(\ldots, v', v, \ldots)\ ,$$

where $|v|$ is a degree of $v \in V$.

n-brackets fulfil homotopy relations:

$$\begin{aligned}
n =1: &\quad \ell_1(\ell_1(v)) = 0, \quad \text{(V, ℓ_1) is a cochain complex}\ ,\\
n =2: &\quad \ell_1(\ell_2(v_1, v_2)) = \ell_2(\ell_1(v_1), v_2) + (-1)^{|v_1|}\, \ell_2(v_1, \ell_1(v_2)) \ \text{ℓ_1 is a derivation of ℓ_2}\ ,\\
n =3: &\quad \ell_1(\ell_3(v_1, v_2, v_3)) = -\ell_3(\ell_1(v_1), v_2, v_3) - (-1)^{|v_1|}\, \ell_3(v_1, \ell_1(v_2), v_3), \quad \text{Jacobi up to homotopy}\\
&\qquad - (-1)^{|v_1|+|v_2|}\, \ell_3(v_1, v_2, \ell_1(v_3))\\
&\qquad - \ell_2(\ell_2(v_1, v_2), v_3) - (-1)^{(|v_1|+|v_2|)\,|v_3|}\, \ell_2(\ell_2(v_3, v_1), v_2)\\
&\qquad - (-1)^{(|v_2|+|v_3|)\,|v_1|}\, \ell_2(\ell_2(v_2, v_3), v_1)\\
&\ldots
\end{aligned}$$

Cyclic L_∞-algebra: graded symmetric non-degenerated bilinear pairing $\langle -, - \rangle : V \otimes V \to \mathbb{R}$

$$\langle v_0, \ell_n(v_1, v_2, \ldots, v_n)\rangle = (-1)^{n+(|v_0|+|v_n|)\,n+|v_n|\,\sum_{i=0}^{n-1}|v_i|}\, \langle v_n, \ell_n(v_0, v_1, \ldots, v_{n-1})\rangle, \quad n \geq 1.$$

How do we use this in gauge field theories?

Start with $V = V_0 \oplus V_1 \oplus V_2 \oplus V_3$. Then

-gauge parameters $\rho \in V_0$,
-gauge fields $A \in V_1$,
-equations of motion $F_A \in V_2$,
-II Noether identites (relations between EoMs F_A) $\mathrm{d}_A F_A \in V_3$.

$$\text{Gauge transformations: } \delta_\rho A = \ell_1(\rho) + \ell_2(\rho, A) - \frac{1}{2}\ell_3(\rho, A, A) + \ldots$$

$$\text{EoM: } F_A = \ell_1(A) - \frac{1}{2}\ell_2(A, A) - \frac{1}{3!}\ell_3(A, A, A) + \ldots$$

$$\text{Action: } S(A) = \frac{1}{2}\langle A, \ell_1(A)\rangle - \frac{1}{3!}\langle A, \ell_2(A, A)\rangle + \ldots$$

Using the cyclicity of the pairing $\langle\, , \rangle$, the variational principle is easily implemented

$$\delta S(A) = \langle \delta A\, ,\, F_A\rangle \ .$$

Example: $3D$ non-Abelian Chern-Simons theory

We define: $\rho \in V_0$, $A \in V_1$, $F_A \in V_2$ and $\mathrm{d}_A F_A \in V_3$

The non-vanishing ℓ_n brackets are given by:

1-bracket ℓ_1

$$\ell_1(\rho) = \mathrm{d}\rho \ \in \ V_1 \ , \ \ell_1(A) = \mathrm{d}A \ \in \ V_2 \ , \ \ell_1(F_A) = \mathrm{d}F_A \ \in \ V_3 .$$

2-bracket ℓ_2

$$\begin{aligned} &\ell_2(\rho_1, \rho_2) = i[\rho_1, \rho_2], \quad \ell_2(\rho, A) = i[\rho, A], \quad \ell_2(\rho, F_A) = i[\rho, F_A] \\ &\ell_2(A_1, A_2) = i[A_1, A_2], \quad \ell_2(A, F_A) = i[A, F_A]. \end{aligned}$$

These reproduce:

$$\begin{aligned} \delta_\rho A &= \ell_1(\rho) + \ell_2(\rho, A) = \mathrm{d}\rho + i[\rho, A], \\ [\delta_{\rho_1}, \delta_{\rho_2}] &= \delta_{-\ell_2(\rho_1,\rho_2)} = \delta_{-i[\rho_1,\rho_2]} \ , \\ F_A &= \ell_1(A) - \frac{1}{2}\,\ell_2(A, A) = \mathrm{d}A - \frac{i}{2}\,[A, A], \\ \delta_\rho F_A &= \ell_2(\rho, F_A) = i[\rho, F_A], \\ \mathrm{d}_A F_A &= \ell_1(F_A) - \ell_2(A, F_A) = \mathrm{d}F_A - \frac{i}{2}\,[A, F_A], \\ S &= \frac{1}{2}\langle A, \ell_1(A)\rangle - \frac{1}{3!}\langle A, \ell_2(A, A)\rangle = \frac{1}{2}\int_M \mathrm{Tr}\Big(A \wedge \mathrm{d}A - \frac{i}{3}\,A \wedge [A, A]\Big). \end{aligned}$$

Braided L_∞-algebra

Rigorously: A braided L_∞-algebra is an L_∞-algebra $(V, \{\ell_n\})$ in the symmetric monoidal category ${}_{\mathcal{F}}\mathcal{M}^\sharp$. What does it means, how does it work?

- $\mathbb{Z}$-graded real vector space $V = \bigoplus_{k\in\mathbb{Z}} V_k$. Usually we work with

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3.$$

- maps/brackets: $\ell_n^\star : \bigotimes^n V \to V$

$$\ell_n^\star(v_1 \otimes \cdots \otimes v_n) = \ell_n(v_1 \otimes_\star \cdots \otimes_\star v_n),$$

with $v \otimes_\star v' := \mathcal{F}^{-1}(v \otimes v') = \bar{\mathrm{f}}^\alpha(v) \otimes \bar{\mathrm{f}}_\alpha(v')$ for $v, v' \in V$. The brackets are graided and braied symmetric!

$$\ell_n^\star(\dots, v, v', \dots) = -(-1)^{|v|\,|v'|}\,\ell_n^\star\big(\dots, \mathrm{R}_k(v'), \mathrm{R}^k(v), \dots\big)\ .$$

For example, in $3D$ CS gauge theory we had $\ell_2(\rho, A) = i[\rho, A]$. This is deformed to

$$\begin{aligned} &\ell_2(\rho, A) = i[\rho, A] \\ &\qquad\downarrow \\ &\ell_2^\star(\rho, A) = i[\bar{\mathrm{f}}^k(\rho), \bar{\mathrm{f}}_k(A)] = i[\rho, A]_\star = -i[\mathrm{R}_k(A), \mathrm{R}^k(\rho)]_\star. \end{aligned}$$

- braided homotopy relations:

$$\ell_1^\star\big(\ell_1^\star(v)\big) = 0 \ ,$$

$$\ell_1^\star\big(\ell_2^\star(v_1, v_2)\big) = \ell_2^\star\big(\ell_1^\star(v_1), v_2\big) + (-1)^{|v_1|}\,\ell_2^\star\big(v_1, \ell_1^\star(v_2)\big) \ ,$$

$$\begin{aligned}
&\ell_2^\star\big(\ell_2^\star(v_1, v_2), v_3\big) - (-1)^{|v_2|\,|v_3|}\,\ell_2^\star\big(\ell_2^\star(v_1, \mathrm{R}_k(v_3)), \mathrm{R}^k(v_2)\big)\\
&\quad + (-1)^{(|v_2|+|v_3|)\,|v_1|}\,\ell_2^\star\big(\ell_2^\star(\mathrm{R}_k(v_2), \mathrm{R}_j(v_3)), \mathrm{R}^j\mathrm{R}^k(v_1)\big)\\
&= -\ell_3^\star\big(\ell_1^\star(v_1), v_2, v_3\big) - (-1)^{|v_1|}\,\ell_3^\star\big(v_1, \ell_1^\star(v_2), v_3\big)\\
&\quad - (-1)^{|v_1|+|v_2|}\,\ell_3^\star\big(v_1, v_2, \ell_1^\star(v_3)\big) - \ell_1^\star\big(\ell_3^\star(v_1, v_2, v_3)\big) \ ,\\
&\dots
\end{aligned}$$

- To have a well defined variational principle, we demand strict cyclicity:

$$\begin{aligned}
&\langle v_2, v_1\rangle_\star = \langle \mathrm{R}_k(v_1), \mathrm{R}^k(v_2)\rangle_\star = \langle v_1, v_2\rangle_\star,\\
&\langle v_0, \ell_n^\star(v_1, v_2, \dots, v_n)\rangle_\star = \langle v_n, \ell_n^\star(v_0, v_1, \dots, v_{n-1})\rangle_\star.
\end{aligned}$$

Twist operator fulfilling this is a compatible Drinfel'd twists. It define a strictly cyclic braided L_∞-algebra.

Braided gauge theory via braided L_∞-algebra

Just like in the classical (commutative) case, a braided L_∞-algebra defines a braided field theory.

Braided gauge transformations

$$\delta^\star_\rho A = \ell^\star_1(\rho) + +\ell^\star_2(\rho, A) - \frac{1}{2}\ell^\star_3(\rho, A, A) + \dots\ .$$

Braided equations of motion

$$\begin{aligned} F^\star_A &= \ell^\star_1(A) - \frac{1}{2}\ell^\star_2(A, A) - \frac{1}{6}\ell^\star_3(A, A, A) + \dots\ , \\ \text{Braided 3D CS:}\quad F^\star_A &= \ell^\star_1(A) - \tfrac{1}{2}\,\ell^\star_2(A, A) = \mathrm{d}A - \frac{i}{2}\,[A, A]_\star\ . \end{aligned}$$

Braided Noether identity does not follow from the variation of an action. Instead it is obtained as a combination of homotopy relations

$$\begin{aligned} \mathsf{d}^\star_A F^\star_A &= \ell^\star_1(F^\star_A) - \tfrac{1}{2}\left(\ell^\star_2(A, F^\star_A) - \ell^\star_2(F^\star_A, A)\right) + \tfrac{1}{4}\,\ell^\star_2\big(\overline{\mathrm{R}}^\alpha(A), \ell^\star_2(\overline{\mathrm{R}}_\alpha(A), A)\big) + \dots \\ &= \mathrm{d}F^\star_A - \frac{i}{2}\,[A, F^\star_A]_\star + \frac{i}{2}\,[F^\star_A, A]_\star + \tfrac{1}{4}\,[\overline{\mathrm{R}}^\alpha(A), [\overline{\mathrm{R}}_\alpha(A), A]_\star]_\star + \dots\ . \end{aligned}$$

Braided gauge invariant action

$$S(A) = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} (-1)^{\frac{1}{2}\, n\,(n-1)} \langle A, \ell_n^\star(A, \dots, A)\rangle \ ,$$

Braided 3D CS:
$$S_\star(A) = \frac{1}{2} \big\langle A, \ell_1^\star(A)\big\rangle_\star - \frac{1}{6} \big\langle A, \ell_2^\star(A, A)\big\rangle_\star$$
$$= \frac{1}{2} \int_M \mathrm{Tr}\Big(A \wedge_\star \mathrm{d}A - \frac{i}{3} A \wedge_\star [A, A]_\star\Big) \ .$$

It is braided gauge invariant $\delta_\rho^\star S_\star(A) = 0$.

Comments on the braided 3D CD theory

-"naive" deformation of the classical theory

-braided II Noether identity: new term (inhomogeneous in EoM), vanishes in the commutative limit. Important, since braided simetries do not act on solutions of EoM in the usual (expected) way:

$$F(A + \delta_\rho^\star A) \neq F(A) + \delta_\rho^\star F(A).$$

Braided gauge symmetries do not transforms (in a usual way) one solution into another.

-more surprises in theories with $\ell_3^\star$ and higher brackets.

Braided 4D ECP gravity

First order formalism: vierbein e and spin connection ω. Symmetries: diffeomorphisms ξ, local Lorentz transformation ρ. Gauge invariant action with a good commutative limit

$$
\begin{aligned}
&S_\star(e,\omega) = \tfrac{1}{2}\left\langle (e,\omega)\,,\,\ell_1^\star(e,\omega)\right\rangle_\star - \tfrac{1}{6}\left\langle (e,\omega)\,,\,\ell_2^\star\big((e,\omega)\,,\,(e,\omega)\big)\right\rangle_\star \\
&-\tfrac{1}{24}\left\langle (e,\omega)\,,\,\ell_3^\star\big((e,\omega)\,,\,(e,\omega)\,,\,(e,\omega)\big)\right\rangle_\star \\
&= \int_M \mathrm{Tr}\Big(\frac{1}{2}\, e \curlywedge_\star e \curlywedge_\star R^\star + \frac{\Lambda}{4}\, e \curlywedge_\star e \curlywedge_\star e \curlywedge_\star e\Big) \\
&-\frac{1}{24}\int_M \mathrm{Tr}\Big(\omega \curlywedge_\star \big(2\, e \curlywedge_\star T_{\mathrm{L}}^\star - 2\, T_{\mathrm{R}}^\star \curlywedge_\star e + \mathrm{d}^\omega_{\star\mathrm{L}}(e \curlywedge_\star e) + \mathrm{d}^\omega_{\star\mathrm{R}}(e \curlywedge_\star e)\big)\Big)\ .
\end{aligned}
$$

It represents a new NC deformation of the ECP gravity (GR).

Covariant equations of motion

$$
F^\star_{(e,\omega)} = \ -\tfrac{1}{2}\,\ell_2^\star\big((e,\omega)\,,\,(e,\omega)\big) - \tfrac{1}{6}\,\ell_3^\star\big((e,\omega)\,,\,(e,\omega)\,,\,(e,\omega)\big) =: (F^\star_e\,, F^\star_\omega),
$$

$$
\begin{aligned}
&\text{Torsion-free cond.} \qquad F^\star_\omega = \tfrac{1}{6}\,\big(e \curlywedge_\star T^\star_{\mathrm{L}} - T^\star_{\mathrm{R}} \curlywedge_\star e - \mathrm{d}^\omega_{\star\mathrm{L}}(e \curlywedge_\star e) - \mathrm{d}^\omega_{\star\mathrm{R}}(e \curlywedge_\star e)\big), \\
&\text{Einstein eq.} \qquad F^\star_e = \tfrac{1}{6}\,\big(2\, e \curlywedge_\star R^\star + 2\, R^\star \curlywedge_\star e + 6\,\Lambda\, e \curlywedge_\star e \curlywedge_\star e \\
&\qquad\qquad + e \curlywedge_\star \mathrm{d}\omega + \mathrm{d}\omega \curlywedge_\star e + \overline{\mathrm{R}}^\alpha(e) \curlywedge_\star [\overline{\mathrm{R}}_\alpha(\omega), \omega]^\star_{\mathfrak{so}(4)}\big).
\end{aligned}
$$

In the commutative limit they reduce to the usual torsion free condition and the 4D Einstein equation.

Braided 4D Electrodynamics

For simplicity: 4D Minkowski space-time, Moyal-Weyl twist, massive spinor field ψ, $U(1)$ gauge field A_μ.
The infinitesimal $U(1)$ gauge transformation:

$$\delta_\rho \psi = i\rho\psi, \quad \delta_\rho \bar{\psi} = -i\bar{\psi}\rho, \quad \delta_\rho A_\mu = \frac{1}{e}\partial_\mu \rho,$$

with the infinitesimal gauge parameter $\rho(x)$. The action and the corresponding equations of motion:

$$S = \int \mathrm{d}^4 x \left(\bar{\psi} i\gamma^\mu (\partial_\mu \psi - ieA_\mu \psi) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}\right),$$

$$i\gamma^\mu(\partial_\mu \psi - ieA_\mu \psi) = 0, \quad (\partial_\mu \bar{\psi} + ie\bar{\psi} A_\mu)\gamma^\mu = 0, \quad \partial_\mu F^{\mu\nu} + e\bar{\psi}\gamma^\mu \psi = 0.$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\bar{\psi} = \psi^\dagger \gamma^0$ and γ^μ are Dirac gamma matrices. The conserved matter current and the corresponding charge:

$$J^\mu = e\bar{\psi}\gamma^\mu \psi, \quad Q = \int \mathrm{d}\vec{x} J^0 = e \int \mathrm{d}\vec{x} \psi^\dagger \psi.$$

To write the L_∞-algebra of classical electrodynamics in a more elegant way we defined a "master" field $\mathcal{A} \in V_1$ and the corresponding equations of motion $F_{\mathcal{A}} \in V_2$ as

$$\mathcal{A} = \begin{pmatrix} \bar{\psi} \\ \psi \\ A \end{pmatrix}, \quad F_{\mathcal{A}} = \begin{pmatrix} F_{\bar{\psi}} \\ F_{\psi} \\ F_A \end{pmatrix}. \tag{2}$$

The corresponding backets are then

$$\begin{aligned} &\ell_1(\rho) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{e}\partial_\mu \rho \end{pmatrix}, \quad \ell_1(\mathcal{A}) = \begin{pmatrix} i\gamma^\mu \partial_\mu \psi \\ -i\gamma^\mu \partial_\mu \bar{\psi} \\ -\partial_\mu \partial_\nu A^\nu + \partial_\nu \partial^\nu A_\mu \end{pmatrix}, \\ &\ell_1(F_{\mathcal{A}}) = \partial_\mu (F_A)^\mu. \end{aligned} \tag{3}$$

$$\ell_2(\rho, \mathcal{A}) = \begin{pmatrix} -i\bar{\psi}\rho \\ i\rho\psi \\ 0 \end{pmatrix}, \ \ell_2(\mathcal{A}_1, \mathcal{A}_2) = -\frac{1}{2} \begin{pmatrix} \gamma^\mu A_{1\,\mu} \psi_2 + \gamma^\mu A_{2\,\mu} \psi_1 \\ \bar{\psi}_1 \gamma^\mu A_{2\,\mu} + \bar{\psi}_2 \gamma^\mu A_{1\,\mu} \\ e(\bar{\psi}_1 \gamma^\mu \psi_2 + \bar{\psi}_2 \gamma^\mu \psi_1) \end{pmatrix}.$$

Corresponding braided L_∞ algebra (using the Moyal-Weyl twist):

$$\ell_1^\star(\rho) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{e}\partial_\mu\rho \end{pmatrix}, \quad \ell_1^\star(\mathcal{A}) = \begin{pmatrix} i\gamma^\mu\partial_\mu\psi \\ -i\gamma^\mu\partial_\mu\bar{\psi} \\ -\partial_\mu\partial_\nu A^\nu + \partial_\nu\partial^\nu A_\mu \end{pmatrix},$$

$$\ell_1^\star(F^\star_{\mathcal{A}}) = \partial_\mu F^\star_A. \qquad (4)$$

$$\ell_2^\star(\rho,\mathcal{A}) = \begin{pmatrix} -i\mathrm{R}_k(\bar{\psi})\star\mathrm{R}^k(\rho) \\ i\rho\star\psi \\ i[\rho,A]_\star = 0 \end{pmatrix},$$

$$\ell_2^\star(\mathcal{A}_1,\mathcal{A}_2) = -\frac{1}{2}\begin{pmatrix} \gamma^\mu A_{1\,\mu}\star\psi_2 + \mathrm{R}_k\gamma^\mu A_{2\,\mu}\star\mathrm{R}^k\psi_1 \\ \bar{\psi}_1\star\gamma^\mu A_{2\,\mu} + \mathrm{R}_k\bar{\psi}_2\star\gamma^\mu\mathrm{R}^k A_{1\,\mu} \\ e(\bar{\psi}_1\gamma^\mu\star\psi_2 + \mathrm{R}_j\bar{\psi}_2\gamma^\mu\star\mathrm{R}^j\psi_1) \end{pmatrix}.$$

Braided action

$$S = \int \mathrm{d}^4x \left\{-\frac{1}{4}F^{\mu\nu}\star F_{\mu\nu} + \bar{\psi}\star i\gamma^\mu\partial_\mu\psi + \frac{1}{2}\Big(\bar{\psi}\star A_\mu\gamma^\mu\star\psi + \bar{\psi}\star\mathrm{R}_k(A_\mu)\gamma^\mu\star\mathrm{R}^k(\psi)\right\}. \quad (5)$$

Braided equations of motion

$$i\gamma^\mu(\partial_\mu\psi - i\frac{1}{2}(A_\mu\star\psi + \mathrm{R}_k(A_\mu)\star\mathrm{R}^k(\psi)) = 0,$$

$$\partial_\mu F^{\mu\nu} = -\frac{1}{2}(\bar{\psi}\star\gamma^\mu\psi + \mathrm{R}_k\bar{\psi}\star\gamma^\mu\mathrm{R}^k\psi).$$

How do we use the II Noether identity? From previos definitions we find

$$\mathrm{d}(F_A^\star)^\mu + \frac{e}{2}\mathrm{d}\big(\bar\psi\star\gamma^\mu\star\psi + \mathsf{R}_k(\bar\psi)\gamma^\mu\star\mathsf{R}^k(\psi)\big) = 0.$$

If $F_A^\star = 0$, then the metter current $\bar\psi\star\gamma^\mu\star\psi + \mathsf{R}_k(\bar\psi)\gamma^\mu\star\mathsf{R}^k(\psi)$ is conserved. The corresponding conserved charge:

$$Q^\star = e\int \mathrm{d}^3\vec{x}\,\big(\psi^\dagger\star\psi + \mathsf{R}_k(\psi^\dagger)\star\mathsf{R}^k(\psi)\big). \tag{6}$$

Nontrivial contribution if $\theta^{0j}\neq 0$.

Quantization? Braided QED is still an abelian gauge theory, no photon self-interaction! Work in progress. . .

Outlook

- We deformed the L_∞-algebra to a braided L_∞-algebra (mathematically well defined in a proper category).

  -well defined way to construct a braided L_∞-algebra starting from the classical one.

  -enables constructions of new NC field theories (unexpected deformations, different from the "naive" expectations).

- Braided Electrodynamics

  -remains $U(1)$ (abelian) gauge theory

  -new term in the action, conserved charge, quantization. . .

- Braided NC gravity

  -braided symmetries close in the braided Lie algebra (no need for the UEA and no new degrees of freedom).

  -unexpected deformation in 4D.

- Future work

  -better understanding of braided symmetries and classical braided field theories (gravity...)

  -quantization of braided field theories