4D NC Gravity from 5D NC CS Theory

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Overview

1. Chern-Simons (CS) Lagrangian
2. NC Field Theory
3. Dimensional (KK) Reduction
4. Results

Introduction

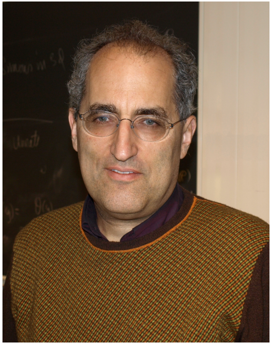


Figure: Image source: Wikipedia

Ingredients

- Lie Group G
- Invariant symmetric form (usually (symmetrised) Trace) on its Lie algebra

$$\left\langle X_1, ..., X_i, ..., X_j, ..., X_n \right\rangle_n = \left\langle X_1, ..., X_j, ..., X_i, ..., X_n \right\rangle_n$$

$$\left\langle X_1, ..., X_n \right\rangle_n = \left\langle U X_1 U^{-1}, ..., U X_n U^{-1} \right\rangle_n, \quad U \in G$$

- Connection A on a principal bundle (*aka* Gauge potential)

In our work, we assume that the bundle is trivial ($\Rightarrow A = A_\mu^A T_A \mathrm{d}x^\mu$).

Chern-Weil Theorem

Curvature $F = \mathrm{d}A + A \wedge A$

Theorem (Chern-Weil):

- $$\mathrm{d}\big\langle F, ..., F \big\rangle_n = 0$$
- $$\big\langle \tilde{F}, ..., \tilde{F} \big\rangle_n - \big\langle F, ..., F \big\rangle_n = \mathrm{d}Q(\tilde{A}, A)$$

We can use $Q(A)$, properly normalised, as our Lagrangian density.

AdS CS Theory

- We will use the Trace in a certain representation of $SO(4,2)$ algebra.

We use generators for $SO(4,2)$ algebra (signautre $(-,+,+,+,+,-)$) as:

$$J_{AB} = \frac{1}{4}[\Gamma_A, \Gamma_B], \qquad A = 0,1,2,3,4;$$

$$J_{A5} = \frac{1}{2}\Gamma_A,$$

where Γ_A are five-dimensional gamma matrices satisfying Clifford algebra $\left\{\Gamma_A, \Gamma_B\right\} = 2G_{AB}$. In terms of four-dimensional gamma matrices they can be written as:

$$\Gamma_a = -i\gamma_a, \qquad a = 0,1,2,3;$$

$$\Gamma_4 = \gamma_5.$$

Note that γ matrices written here are those for a different metric signature $(+,-,-,-)$ (and thus the imaginary unit in front).

AdS CS Theory

- Take $A = \frac{1}{2}\omega^{AB} J_{AB} + \frac{1}{l} e^A J_{A5}$, where ω^{AB} is interpreted as spin connection and e^A as vierbein
- We have to set normalisation:

$$\frac{-i}{3}\kappa Tr(F \wedge F \wedge F) = \mathrm{d}L^{(5)}_{CS}$$

Compute: $L^5_{CS} = \frac{\kappa}{8}\varepsilon_{ABCDE}\Big(\frac{1}{5l^5} e^A e^B e^C e^D e^E + \frac{2}{3l^3} R^{AB} e^C e^D e^E + \frac{1}{l} R^{AB} R^{CD} e^E\Big)$.
More falimilar form:

$$\frac{\kappa}{2l^3}\int \mathrm{d}^5 x\,\sqrt{-g}\Big(R + \frac{6}{l^2} + \frac{l^2}{4}(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})\Big)$$

NC Field Theory

- Standard picture: $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, $\theta^{\mu\nu}$ is a constant antisymmetric matrix
- We deform algebra of function on a commutative space-time

$$f \star g = e^{\frac{i}{2}\theta^{\mu\nu}\partial_{x^\mu}\partial_{y^\nu}} f(x)g(y)|_{x\to y} \tag{1}$$

- In general not clear how to choose a coordinate system (why would we have to choose it?)
- Generalisation: abelian Drinfeld Twist

$$f \star g = \mu\{\mathcal{F}^{-1} f \otimes g\}, \qquad \mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{AB} X_A \otimes X_B},$$

where X_A are commuting vector fields.
If we choose coordinate vector fields as X_A, we get eq. (1)

NC Gauge Theories

$$\delta_\epsilon \psi = \epsilon\psi \quad \delta_\epsilon A = -\mathrm{D}\epsilon = -\mathrm{d}\epsilon - [A, \epsilon]$$

$$\delta_\epsilon F = [\epsilon, F]$$

- In commutative case we have:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{-[\epsilon_1, \epsilon_2]}.$$

- In noncommutative case one can prove that

$$[\delta^\star_{\hat{\epsilon}_1}, \delta^\star_{\hat{\epsilon}_2}]_\star = -\frac{1}{2}\Big(\{\hat{\epsilon}_1^A, \hat{\epsilon}_2^B\}_\star [T_A, T_B] + [\hat{\epsilon}_1^A, \hat{\epsilon}_2^B]_\star \{T_A, T_B\}\Big)$$

NC Gauge Theories

- Solution: Enveloping algebra approach.
- Seiberg-Witten map: NC fields are related to odinary (commutative) fields

$$\hat{A}(A + \delta_\epsilon A) = \hat{A}(A) + \delta_{SW}\hat{A}(A)$$

$$\delta_{SW}\hat{A} = \hat{\delta}_{\hat{\epsilon}}\hat{A} = -\mathrm{d}\hat{\epsilon} + \hat{\epsilon} \wedge_\star \hat{A} - \hat{A} \wedge_\star \hat{\epsilon}$$

- From this we can calculate in the first order in θ^{AB} correction to A

$$\hat{A} = A - \frac{i}{4}\theta^{IJ}\{A_I, \ell_J A + F_J\}$$

- We can also define star wedge product using Lie derivatives, similar as before.

NC Correction to 5D AdS CS

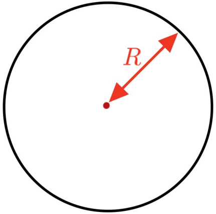
Starting from $\frac{-i}{3}\kappa Tr(F \wedge^\star F \wedge^\star F)$ one can derive the the first order correction do 5D AdS CS action :

$$
\begin{aligned}
S_{NC}^{(1)} = \frac{\kappa}{12}\theta^{IJ}\int \Big(& F^{AB}(F_I)_{BC}(\mathrm{D}_\omega F_J)^C{}_A + \frac{1}{l^2}F_A{}^B(F_I)_{BC}(T_J)^C e^A \\
& + \frac{1}{l^2}F^{AB}(T_I)_B(\mathrm{D}_\omega T_J)_A + \frac{2}{l^2}F^{AB}(T_I)_B(F_J)_{AC}e^C \\
& + \frac{1}{l^2}T^A(T_I)^B(\mathrm{D}_\omega F_J)_{BA} + \frac{1}{l^2}T^A(\mathrm{D}_\omega T_I)^B(F_J)_{BA} \\
& + \frac{1}{l^2}T_A(F_I)^{AB}(F_J)_{BC}e^C + \frac{2}{l^4}T_A(T_I)_B(T_J)^{[B}e^{A]}\Big).
\end{aligned}
$$

[Aschieri,Castellani]

Kalutza Klein Idea

- Original idea: gravity in five dimensions gives, upon compactifying the additional dimension, gravity in four dimensions plus a $U(1)$ gauge field plus scalar (dilaton/radion)
- Reduction from five to four dimensions: compactify on S^1
- In general, fields are periodic in coordinate S^1, so we use Fourier expansion
- This gives rise to an infinite tower of massive states, that are suppressed by a factor $\frac{1}{R}$



Dimensional Reduction

- In our work we will focus solely on a gravity part; we will take all fields independent of the coordinate on S^1 and will truncate other fields (apart from gravity part and a scalar) in a way to preserve $SO(3,2)$ symmetry.
- Starting from AdS CS action in five dimensions, one obtains [Chamseddine]

$$\frac{\kappa\pi R}{4l^2}\int \varepsilon_{ABCDE}\phi^A F^{BC} F^{DE},$$

where we set $\phi^a = (\omega_4^{4a}, e_4^4)$, and truncated fields as promised before.

Symmetry Breaking

- This Lagrangian has $SO(3,2)$ gauge symmetry.
- By choosing $(\phi^A = (0,0,0,0,l))$ we get EH Lagrangian with cosmological constant

$$\frac{\kappa\pi R}{8l}\int \varepsilon_{abcd}\left(R^{ab}R^{cd} - \frac{2}{l^2}R^{ab}e^c e^d + \frac{1}{l^4}e^a e^b e^c e^d\right)$$

$4D$ Noncommutativity

- First idea: restrict noncommutitvity to reduced $4D$ space-time.
- Do the reduction of the first order correction using that $X_I^4 = 0$.
- Result is **zero**
- In accordance with [Ulas Saka,Uler] and [Dimitrijevic,Radovanovic]

Noncommutativity with the extra dimension

- By taking X_I^4 different from zero, we get many surviving terms
- Luckily, after symmetry breaking, and dropping terms containing ∂X_I, we are left with only four terms

$$\boxed{-\frac{\kappa\pi R}{6}\theta^{IJ}X_I^{\alpha}X_J^4\varepsilon^{\mu\nu\rho\sigma}\left(\frac{1}{2l^4}R_{\mu\nu}^{ab}T_{\rho\sigma a}e_{\alpha b}-\frac{2}{l^4}T_{\mu\nu}^{a}R_{\alpha\rho ab}e_{\sigma}^{b}+\frac{1}{l^4}R_{\mu\nu}^{ab}T_{\alpha\rho a}e_{\sigma b}+\frac{3}{l^6}T_{\mu\nu}^{a}e_{\rho a}g_{\alpha\sigma}\right)}$$

Equations of motion

- In the commutative limit we have

$$\varepsilon_{abcd} R^{ab} e^c = 0, \qquad \varepsilon_{abcd} T^a e^b = 0$$

- We can write down general variation of our NC correction, and obtain equations of motion in general.
- As in commutative limit torsion is zero, and we will work perturbatively in θ^{IJ}, we will vary only torsion in this part of the action.
- One can then get variations for vielbein fields and for spin connection:

$$-\frac{\kappa\pi R}{6}\theta^{IJ} X_I^{\alpha} X_J^4 \varepsilon^{\mu\nu\rho\sigma}\left(-\frac{4}{l^4}(\mathrm{D}_\mu R_{\alpha\rho ab}) e_\nu^b - \frac{1}{l^4} R_{\mu\nu ab}\mathrm{D}_\rho e_\alpha^b + \frac{6}{l^6}(\mathrm{D}_\mu e_{\alpha b}) e_\nu^b\right)\delta e_\sigma^a$$

$$-\frac{\kappa\pi R}{6}\varepsilon^{\mu\nu\rho\sigma}\left(-\frac{1}{l^4} R_{\rho\nu ab} e_\alpha^c - \frac{4}{l^4} e_\nu^c R_{\alpha\rho ab} e_\sigma^b + \frac{6}{l^6} e_\nu^c e_{\rho a} g_{\alpha\sigma}\right)\delta\omega_{\mu c}^a$$

AdS Solution

- We start from the solution of commutative part (AdS solution) $\mathrm{d}s^2 = -\left(1+\frac{r^2}{l^2}\right)\mathrm{d}t^2 + \frac{1}{\left(1+\frac{r^2}{l^2}\right)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$
- $R^{ab} = -\frac{1}{l^2}e^a e^b$
- Putting into our equation of motion, we see that this solution is still the solution, even after including the first order correction.
- Next move: $\mathrm{d}s^2 = -\left(1+\frac{r^2}{l^2}-\frac{2m}{r}\right)\mathrm{d}t^2 + \frac{1}{\left(1+\frac{r^2}{l^2}-\frac{2m}{r}\right)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$ (work in progress!)

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