Entropy of Hawking radiation

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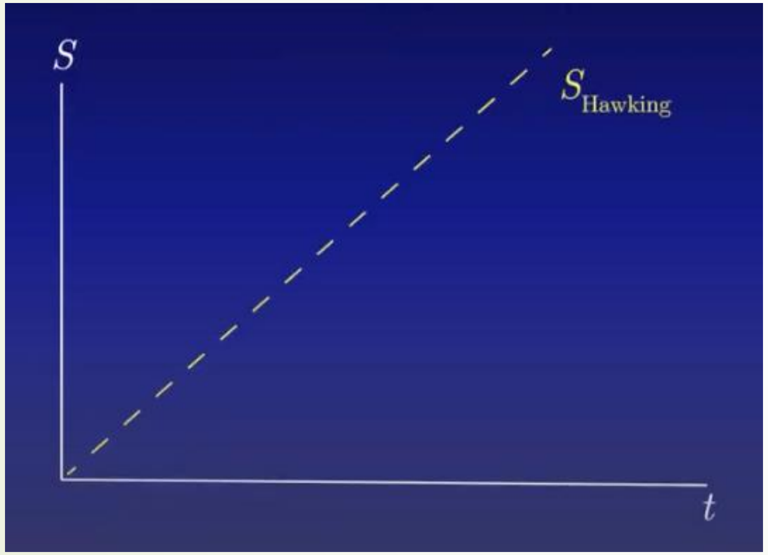
A little bit of history

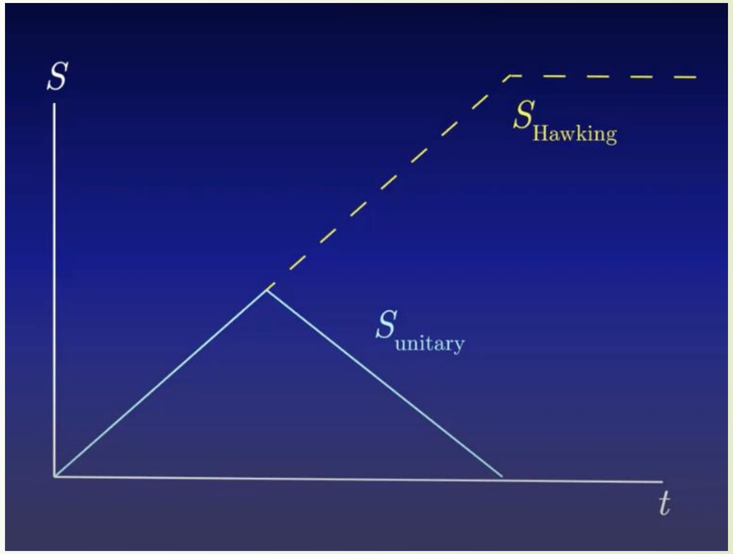
- 1974. Hawking discovered that black holes emit radiation. The main problem was that his calculation was in dispute with quantum mechanics' unitary evolution. Hence we have information loss paradox.
- 1992. Don Page proposes a curve that entropy should follow if the evolution was indeed unitary. Many attempts were made in the following years to reproduce that curve. They were unsuccessful until late 2010s
- During the nineties many toy models were created to study black hole evaporation. Most prominent being JT and RST/BPP models. In JT model Page's curve was, finally, reproduced in 2019, while in the RST/BPP it was reproduced in 2020-2021 with help of a new formula for fine-grained entropy in gravitational systems developed by Maldecena in 2013.

Information paradox

- Hawking's curve:



- Page's curve:



Replica wormholes

- A method for calculating fine-grained entropy in gravitational systems based on euclidean gravitational path integral:

$$\mathcal{Z} = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\phi e^{-S_E[g_{\mu\nu},\phi]} = e^{-S_E\left[g_{\mu\nu}^{(0)},\phi^{(0)}\right]}$$

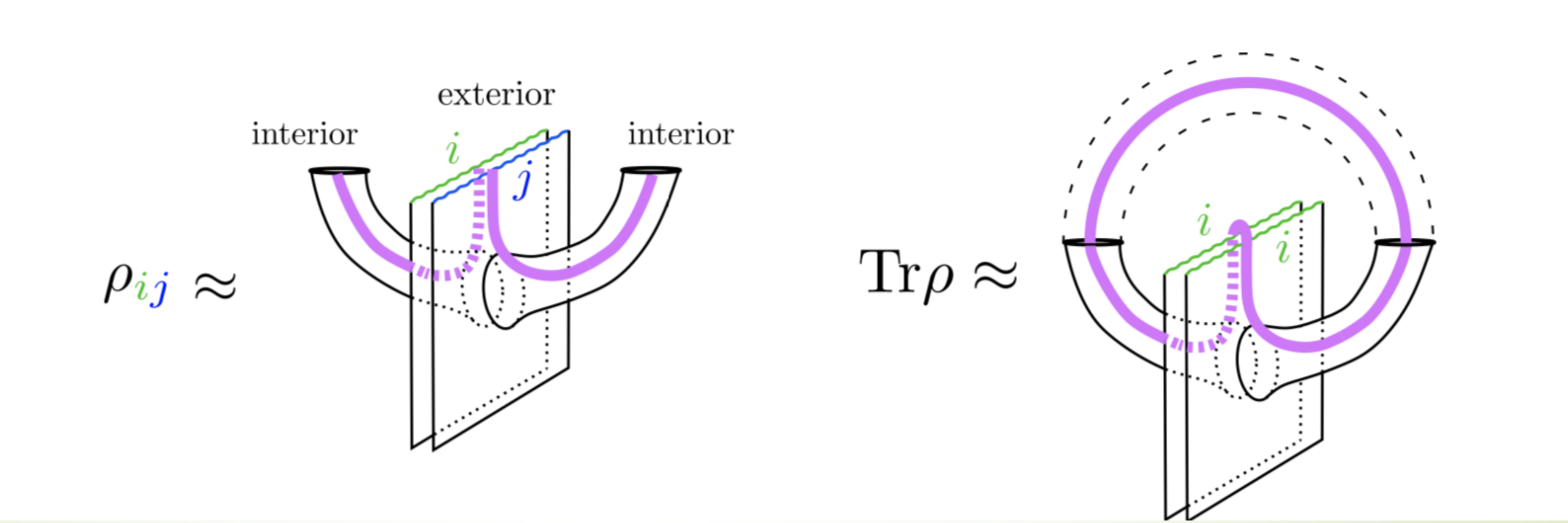
- It is a method developed by Hawking. He used the exact same method to derive generalized entropy formula as course-grained entropy during the seventies:

$$S_{gen} = \frac{A}{4G} + S_{outside}$$

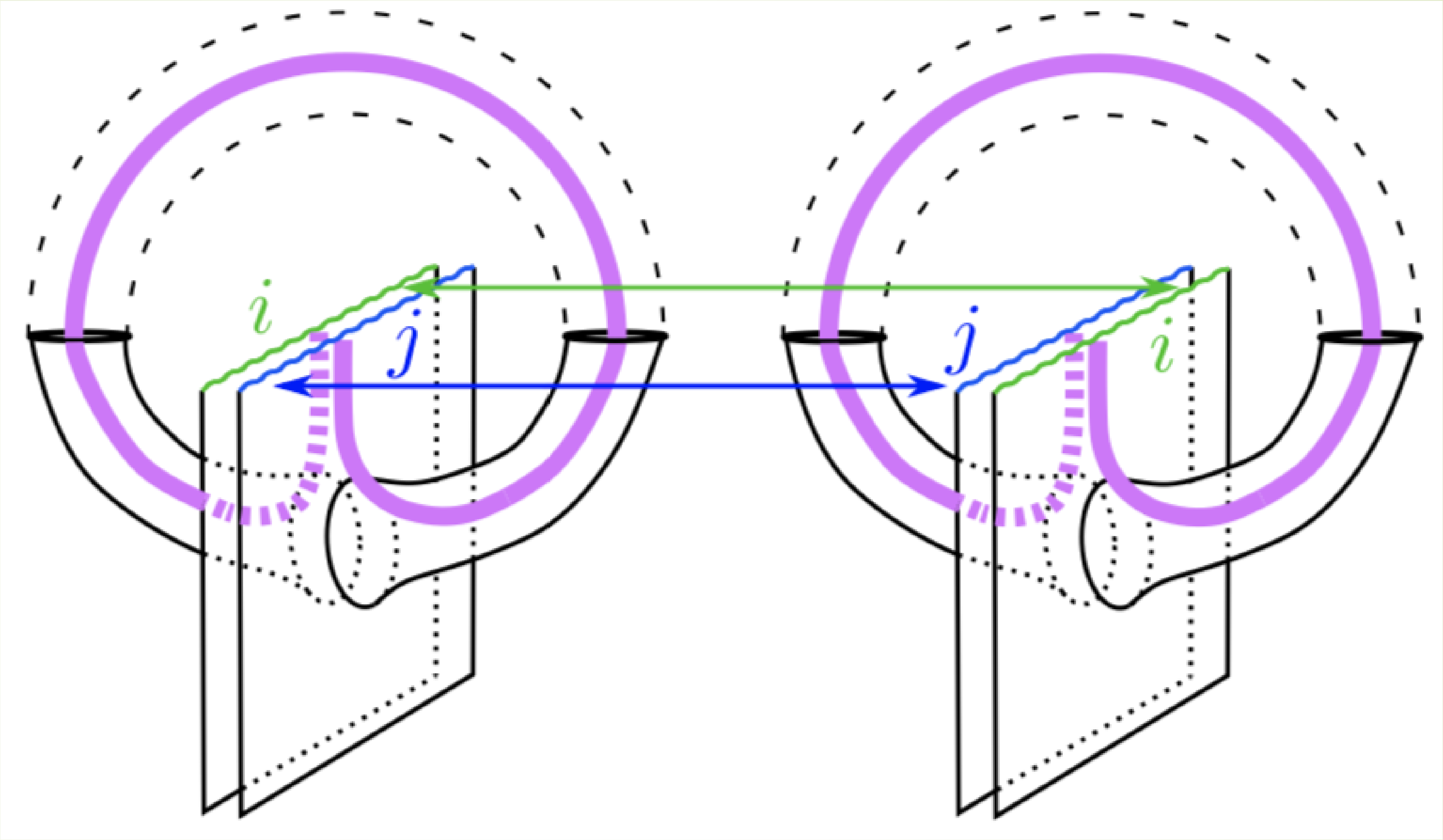
Replica wormholes

- Hawking's calculation for the fine-grained entropy of quantum fileds, outside the black hole, results into fine-grained entropy as always increasing function of time. That is why we have information paradox!
- What is wrong with Hawking's calculation?

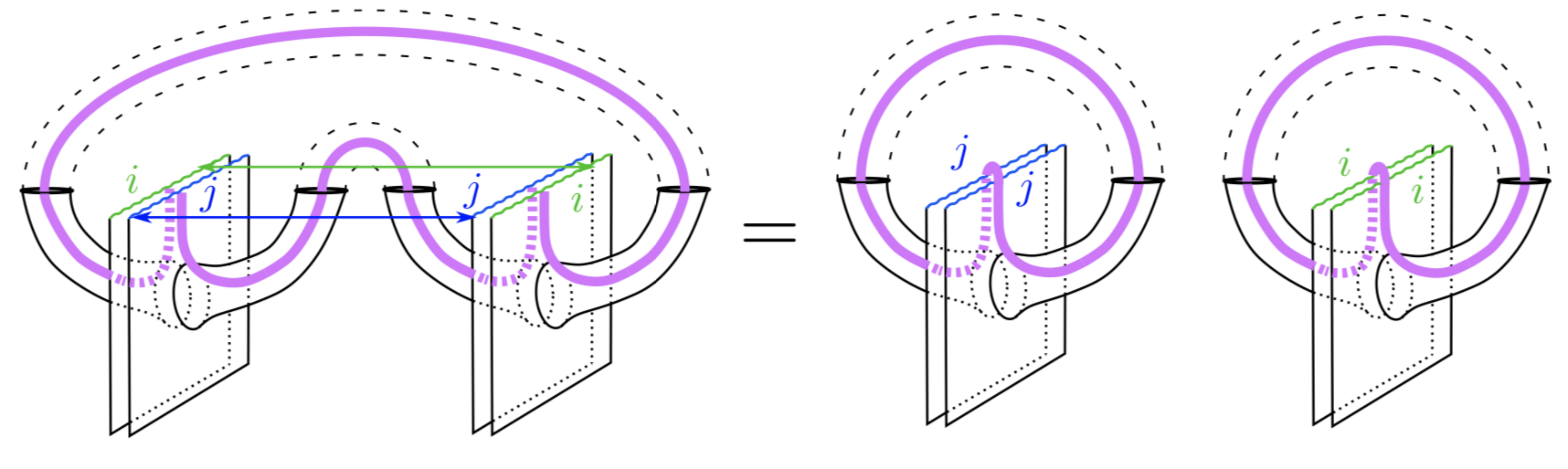
  Answer: He used wrong saddle point in gravitational path integral!
- When you consider summing over topologies, you get two saddle points in gravitational path integral. One correspondes to Hawking's result, and the other coresponds to replica wormholes result. The one that is a global minimum is used when calculating path integral. Over time the global minimum changes between these two saddle points. That point in time coresponds to a phase transition which occures at Page's time.



Let's say that singularity formed from a pure state $|\psi\rangle$. Then we have density matrix: $\rho = |\psi\rangle\langle\psi|$. We have a basis of final states of radiation after the black hole has evaporated given by: $\{|i\rangle\}$. In the picture above we have represented matrix elements of density matrix in this final moment of evaporation, as well as the trace of this density matrix. What we are interested in is if that final state is pure or mixed. Than can be checked using usual purity condition: $\mathrm{Tr}\{\rho^2\} = (\mathrm{Tr}\{\rho\})^2$. We can check this using diagrams as well!



Now we have two copies of matrix elements: $\mathrm{Tr}\{\rho^2\} = \sum_{i,j} \rho_{ij}\rho_{ji}$. We need to connect interiors. In this case we can do that in two ways. This is one of them. It is obvious that here we have only one closed loop. So this does not corespond to the pure state.



In this case it is obvious that the final state is pure. This case coresponds to replica wormholes saddle point, while the previous one coresponds to Hawking's saddle point. That means that the minimum at the end of the evaporation should be given by the replica wormholes' saddle point!

Replica trick and fine-grained entropy

- Using simmilar replica trick we can calculate fine-grained entropy:

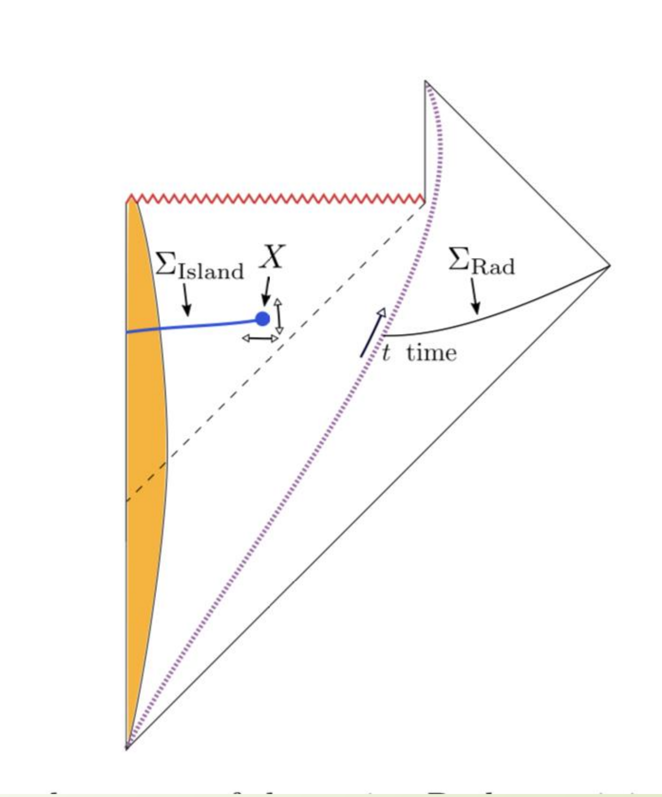
$$S_{FG} = (1 - n\partial_n) \ln \left(\mathrm{Tr}\{\rho^n\}\right), \qquad n \to 1$$

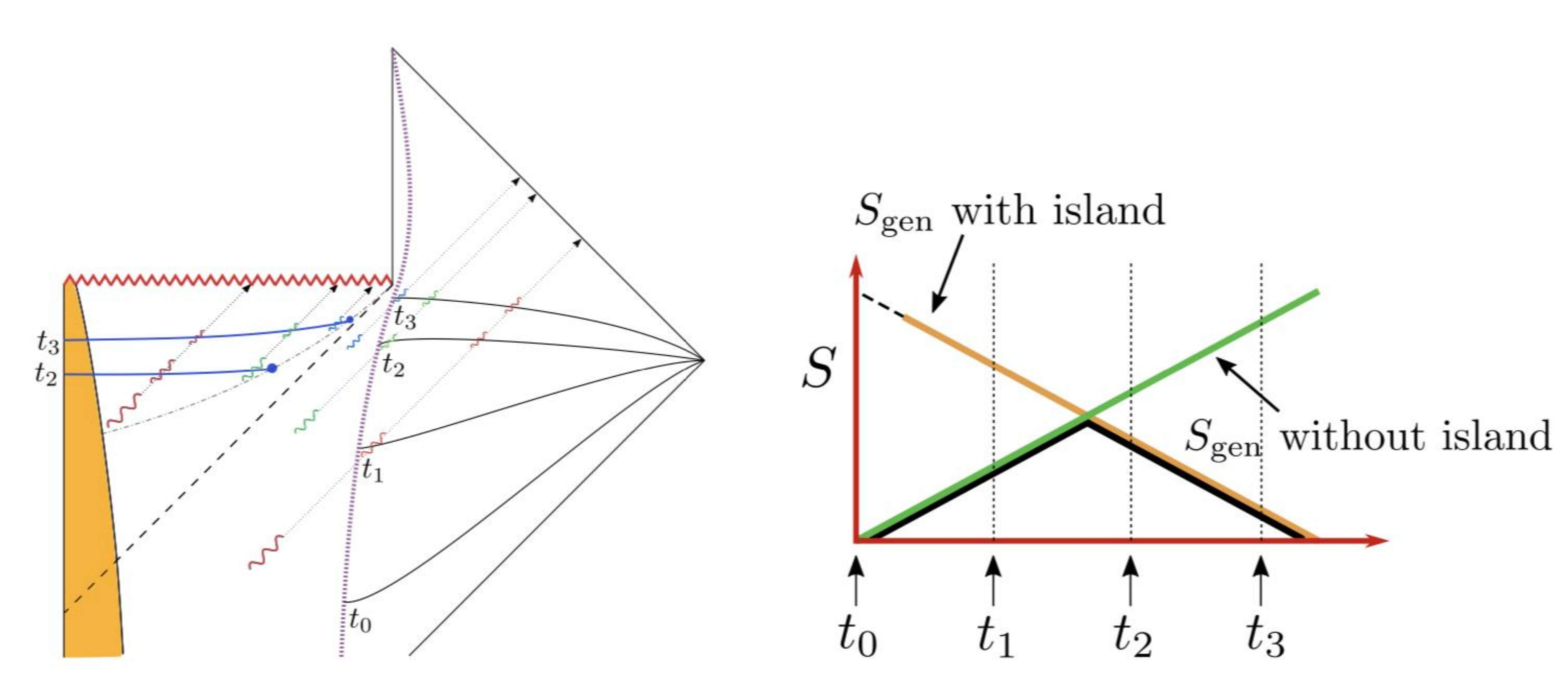
- For arbitrary n, we have many different ways to connect the interiors, so the calculation is much more complicated. When you use this trick you get fine-grained formula for gravitational systems given by:

$$S_{FG} = \min_I \left\{ \operatorname*{ext}_I \left[\frac{A[I]}{4G} + S_{semi-cl} \left(\Sigma_I \cup \Sigma_{rad} \right) \right] \right\}$$

Island rule

- Object *I* that with respact to which we minimize expression in the formula for fine-grained entropy is called an island. It is a codimension 2 surfice (*X* in the picture).
- If there is no island, it is obvious, that we get Hawking's result, but if there is an island, then we get replica wormholes result. That means that during evaporation, sometimes we will have island present, and sometimes we won't.





Island rule and Page's curve

In this picture we see how we reproduce Page's curve using formula for fine-grained entropy in gravitational systems!

BPP model

- This is exactly solvable model of 2D dilaton gravity. The action for classical CGHS model is given by:

$$S_{CGHS} = \frac{1}{2\pi G}\int d^2x\sqrt{-g}e^{-2\phi}\left[\mathcal{R} + 4\left(\nabla\phi\right)^2 + 4\lambda^2\right] - \frac{1}{2}\sum_{i=1}^{N}\int d^2x\sqrt{-g}\left(\nabla f_i\right)^2$$

- To this action we add quantum fluctuations trought Polyakov-Liouville term:

$$S_{PL} = -\frac{\hbar}{96\pi}\int d^2x\int d^2y\sqrt{-g(x)}\sqrt{-g(y)}\mathcal{R}(x)G(x-y)\mathcal{R}(y)$$

- To be exactly solvable we add an additional term to the action:

$$S_{cor} = \frac{N\hbar}{24\pi}\int d^2x\sqrt{-g}\left[-\phi\mathcal{R} + \left(\nabla\phi\right)^2\right]$$

BPP model-equations of motion

- We fix conformal gauge: $ds^2 = -e^{2\rho} dx^+ dx^-$. We can fix Kruskla gauge: $\rho = \phi$ as well. In this gauge, solution to the equations of motion is given by:

$$e^{-2\phi} = -\lambda^2 x^+ x^- - \pi G \int dx^+ \int dx^+ \left(T^{f,cl}_{++} - \varepsilon t_+\right) - \pi G \int dx^- \int dx^- \left(T^{f,cl}_{--} - \varepsilon t_-\right) + a_+ x^+ + a_- x^- + b$$

- There are few solutions that are worth mentioning:

1. Linear dilaton vacuum
2. Static black hole
3. Minkowski vacuum
4. Gravitational collapse scenario

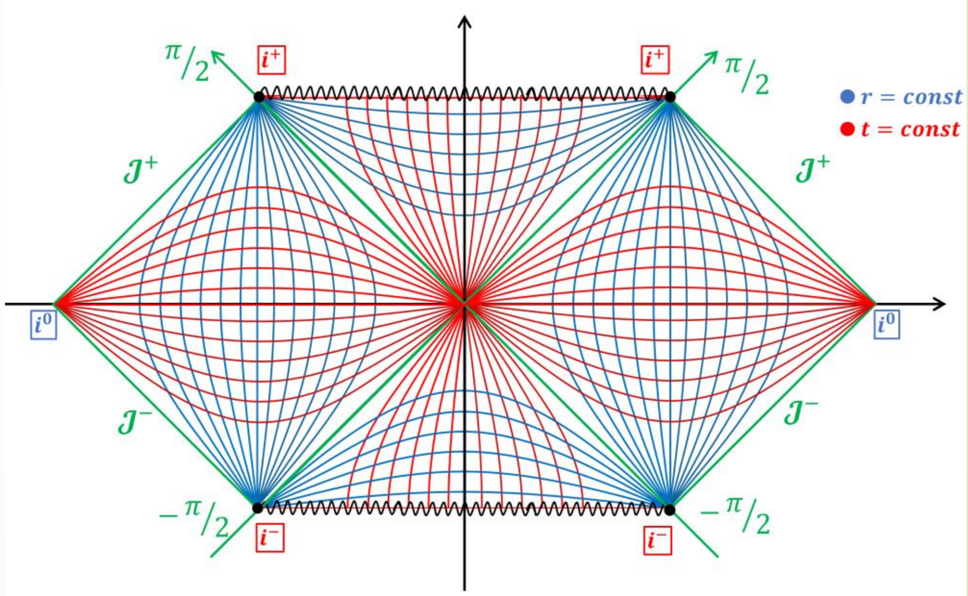
Eternal black hole in BPP

- This is a static black hole solution. The metric is given by:

$$ds^2 = -\frac{dx^+ dx^-}{-\lambda^2 x^+ x^- + \frac{\pi G M}{\lambda}}$$

- On the left side we can see a conformal diagram. It is the same as for the Schwarzschild black hole.

- Conformal diagram:

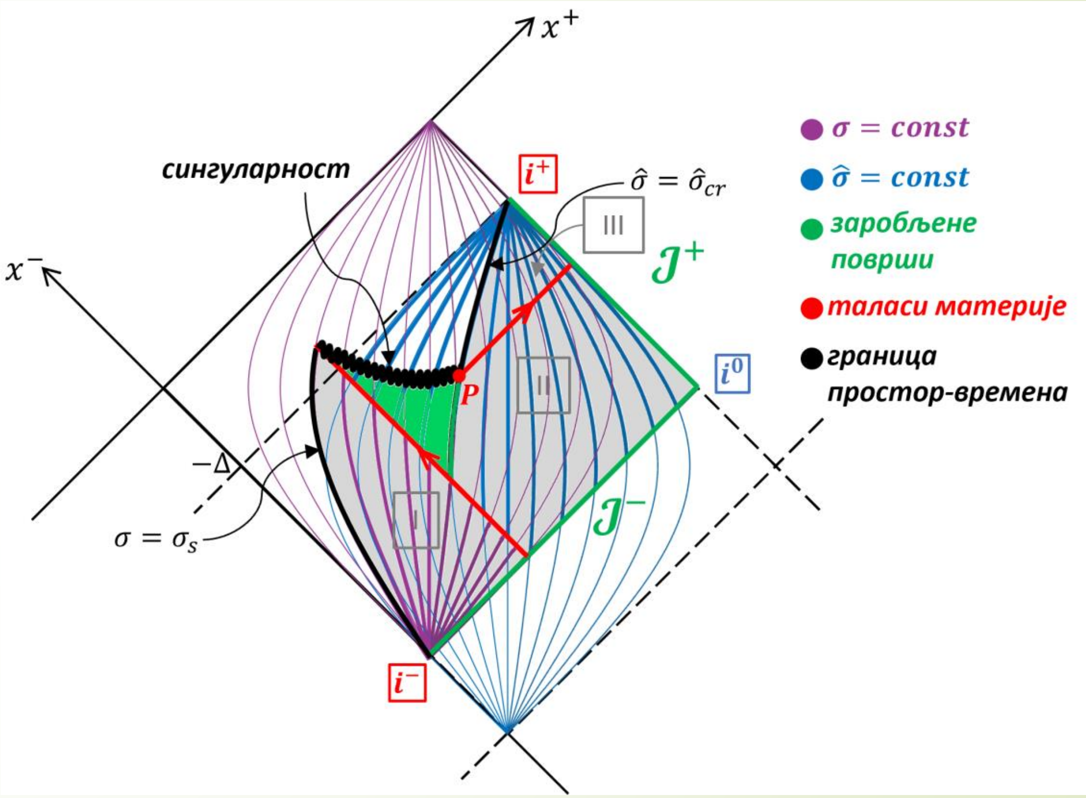


Gravitational collapse scenario in BPP

- This is a dynamic scenario where a singularity is formed from collapsing matter. The metric is given by:

$$ds^2 = \begin{cases} -\dfrac{dx^+dx^-}{-\lambda^2 x^+ x^- - \frac{\varepsilon\pi G}{4}\ln(-\lambda^2 x^+ x^-) + C}, & \text{у области } I \\ -\dfrac{dx^+dx^-}{-\lambda^2 x^+(x^- + \Delta) - \frac{\varepsilon\pi G}{4}\ln(-\lambda^2 x^+ x^-) + \frac{\pi G M}{\lambda} + C}, & \text{у области } II \\ -\dfrac{dx^+dx^-}{-\lambda^2 x^+(x^- + \Delta) - \frac{\varepsilon\pi G}{4}\ln(-\lambda^2 x^+(x^- + \Delta)) + C^*}, & \text{у области } III \end{cases}$$

- On the left side we can see a conformal diagram for this dynamical solution.



Page curve for eternal black hole (BPP)

- First we do the calculation without the island.
- Formula for fine-grained entropy is given by:

$$S_{FG} = \frac{N}{12} \ln \frac{(x_R^+ - x_L^+)^2 (x_R^- - x_L^-)^2}{\delta^4 e^{-2\rho_L} e^{-2\rho_R}}$$

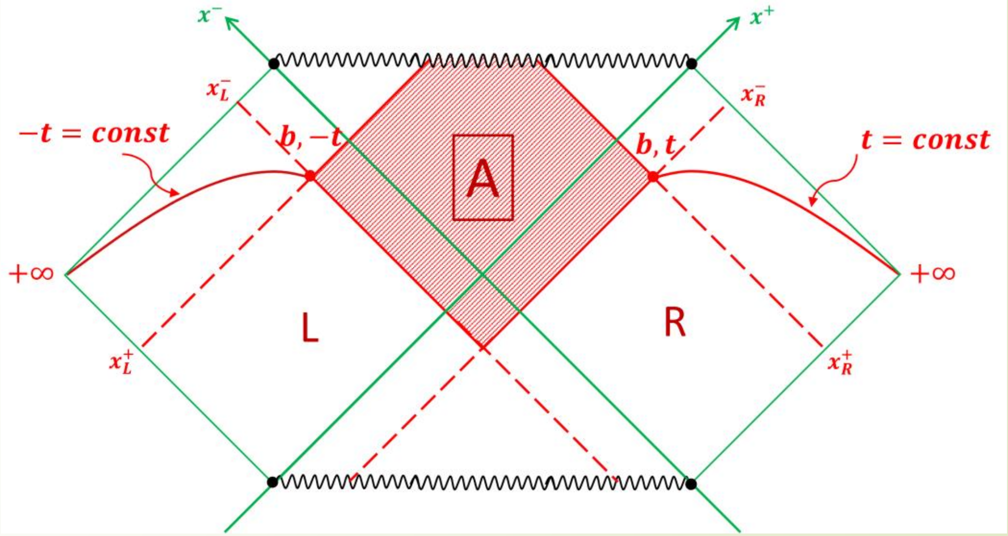
- At late time the dominant term is:

$$S_{FG} = \frac{N}{3} \lambda t$$

- The coordinate transformation is given by:

$$\left|\lambda x^{\pm}\right| = e^{\pm\lambda(t \pm \sigma)}$$

- The situation on the conformal diagram is given by:



- This is Hawking's result, where fine-grained entropy always increases

Page curve for eternal black hole (BPP)

- Now we do calculation with island present.
- The semiclassical entropy is:

$$S_{semi-cl} = \frac{N}{6} \ln \frac{d_{12}^2 d_{23}^2 d_{14}^2 d_{34}^2}{\delta^4 d_{24}^2 d_{13}^2 e^{-\rho_1} e^{-\rho_2} e^{-\rho_3} e^{-\rho_4}}$$

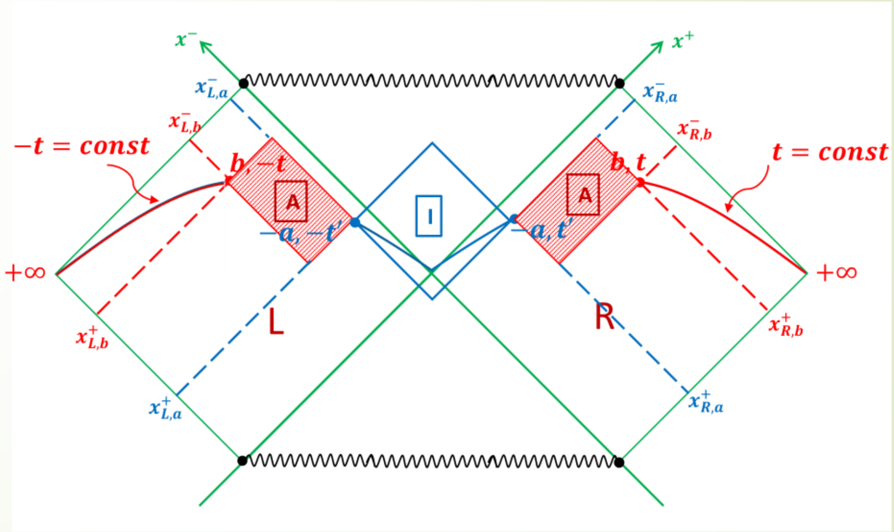
$$d_{ij}^2 = (x_i^+ - x_j^+)(x_i^- - x_j^-)$$

- Extremizing with respect to t' and a, we find that the minimum is given by $t' = t$ and:

$$e^{\lambda a_\pm} = \frac{6}{N}\left[1 \pm \sqrt{1 - \frac{N}{3} e^{-2\lambda b}}\right]$$

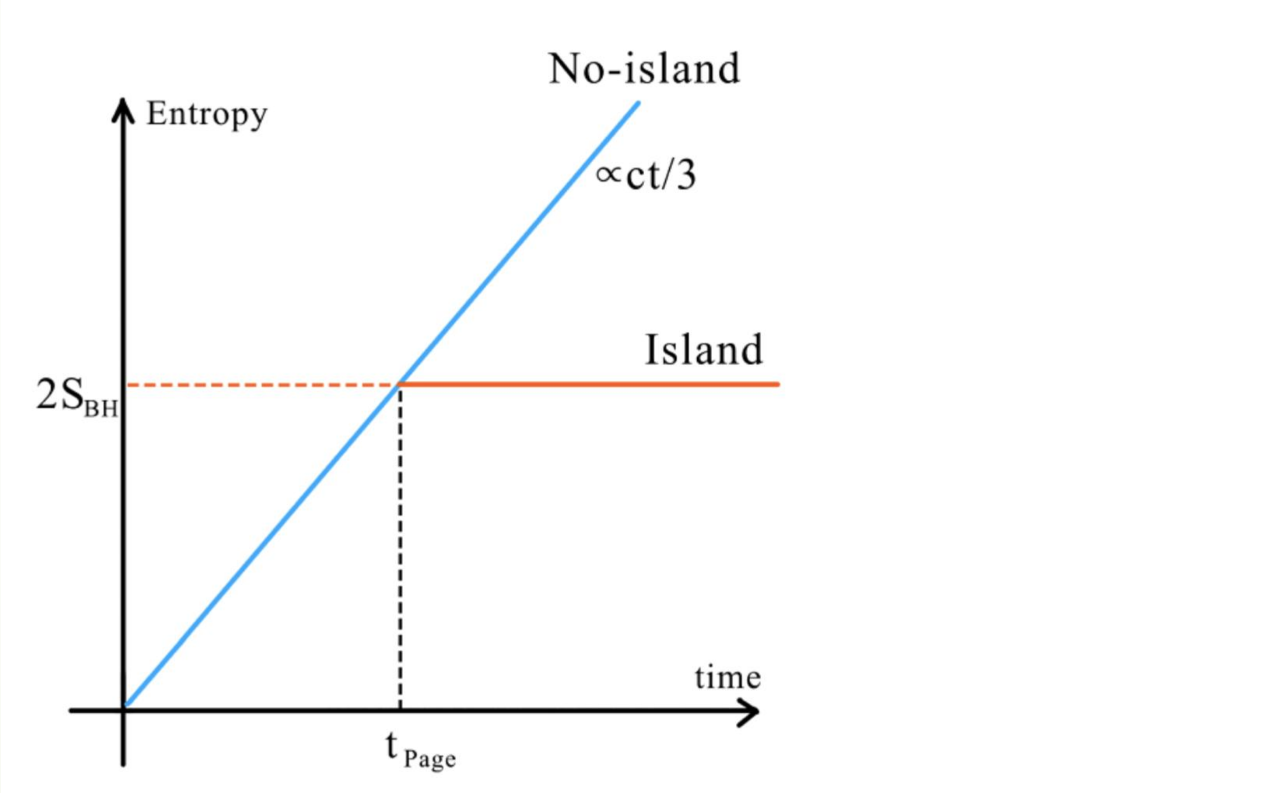
- The global minimum is given by the plus sign
- For the gravitational part we have:

$$\frac{A[I]}{4G} = 2\left(2e^{-2\phi(a)} - \frac{N}{6}\phi(a)\right)$$



- The entropy at late time becomes constant: $S = 2S_{BH}$!

Thus we have reproduced the Page curve!



Page curve for evaporating black hole

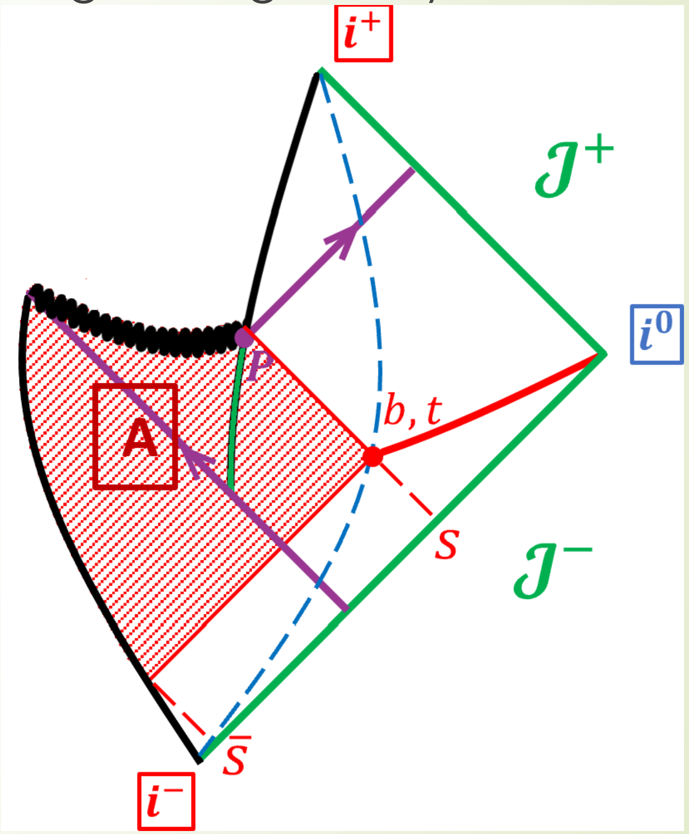
- Once again we start with no island calculation. The entropy is given by: $$S_{FG} = \frac{N}{12} \ln \frac{(\sigma_S^+ - \sigma_{\bar{S}}^+)^2}{\delta^2 e^{-2\rho_s(\sigma)}}$$
- At late time the dominant term is:

$$S_{FG} = \frac{N}{12} \lambda t$$

- The coordinate transformation is given by:

$$\lambda x^+ = e^{\lambda(t+\hat{\sigma})}$$
$$\lambda(x^- + \Delta) = -e^{-\lambda(t-\hat{\sigma})}$$

- We need to write σ in terms of $\hat{\sigma}$ coordinates!
- The situation on the conformal diagram is given by:



- We see that we have Hawking's result once more

Page curve for evaporating black hole

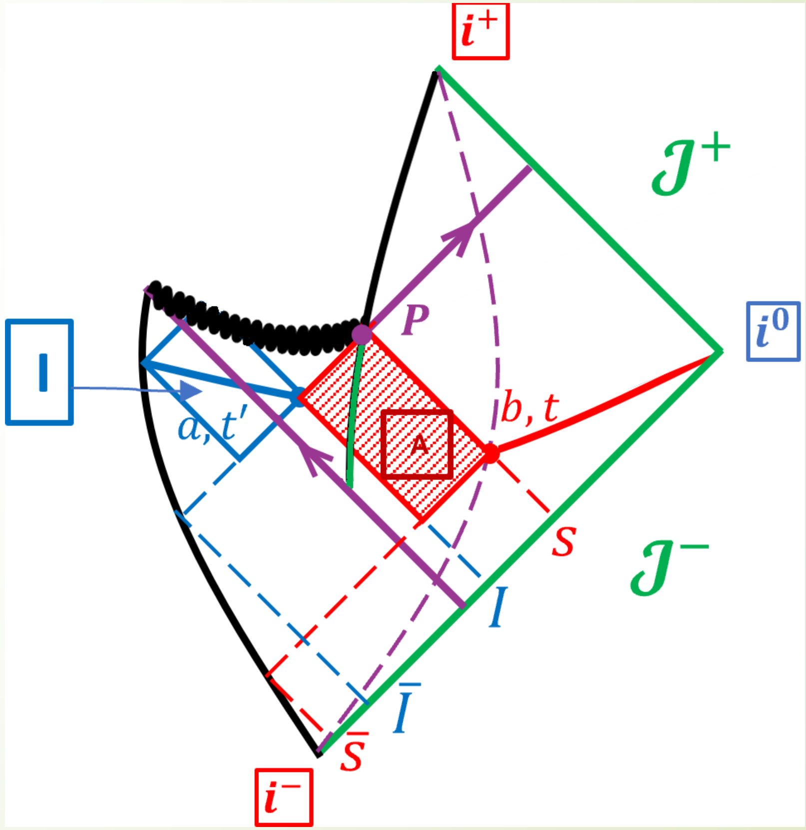
- Now we consider an island rule entropy. A formula for semiclassical entropy is complicated in this case, so we are not going to present it here.
- After the calculation, the position of the island is given by:

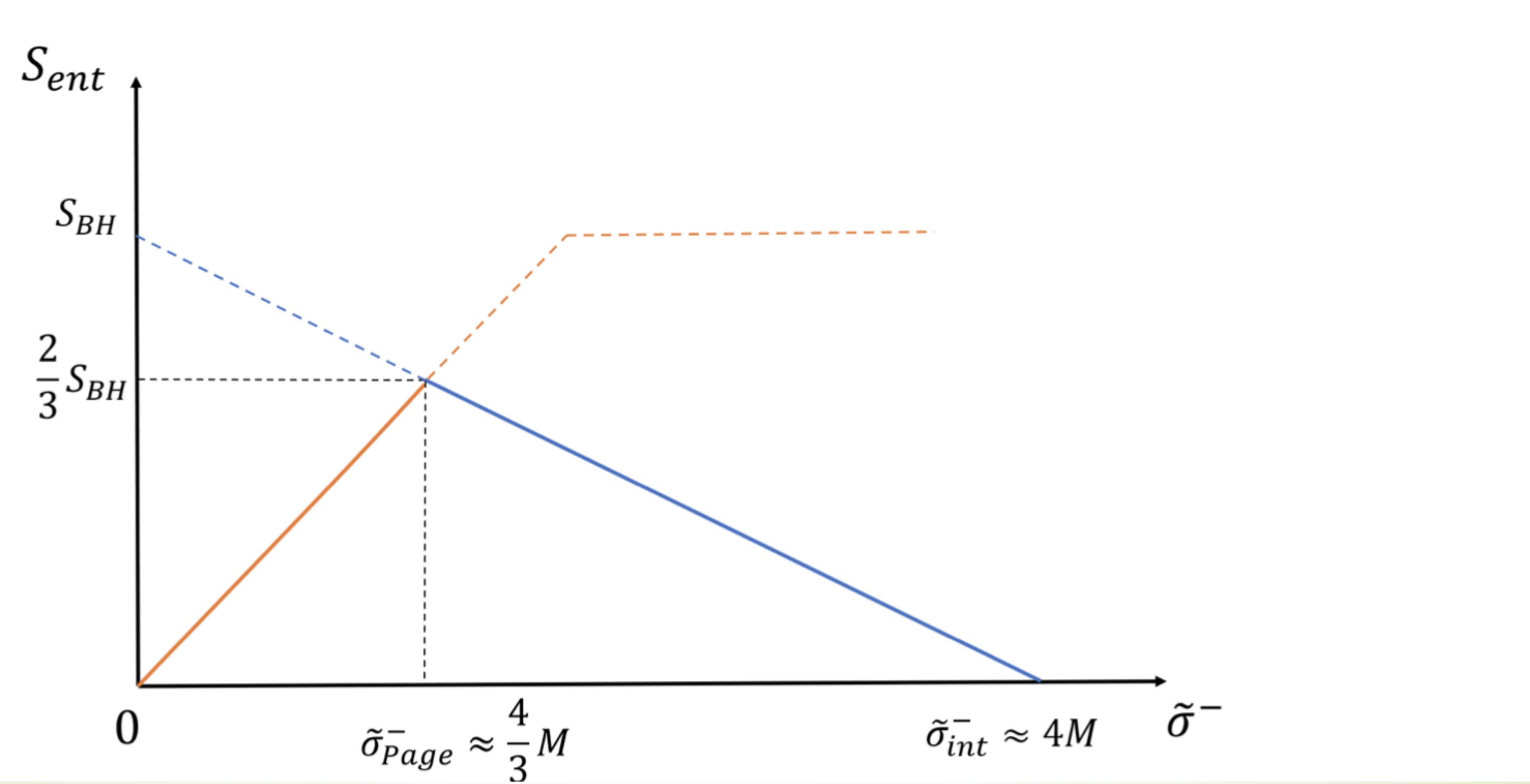
$$\lambda^2 x_I^+ (x_I^- + \Delta) = \frac{\varepsilon \pi G}{4}$$

- The entropy at late time becomes:

$$S_{FG} = S_{BH} - \frac{N}{24}\lambda t$$

- The situation on the conformal diagram is given by:





Page curve for evaporating black hole in BPP model

We have once again reproduced the Page curve. Fine-grained entropy is given by: $S_{FG} = \min\left\{\frac{N}{12}\lambda t, S_{BH} - \frac{N}{24}\lambda t\right\}$ From this we see when the phase transition occures, and get the Page time.

Dimensional reduction of Einstein-Hilbert action

- After dimensional reduction action becomes:

$$S = \frac{1}{4G}\int d^2x\sqrt{-g}e^{-2\phi}\left[\mathcal{R} + 2(\nabla\phi)^2 + 2\lambda^2 e^{2\phi}\right] - \frac{1}{2}\sum_{i=1}^{N}\int d^2x\sqrt{-g}\,(\nabla f_i)^2$$

- We again add Polyakov-Liouville term to get quantum fluctuations:

$$S_{PL} = -\frac{\hbar}{96\pi}\int d^2x\int d^2y\sqrt{-g(x)}\sqrt{-g(y)}\mathcal{R}(x)G(x-y)\mathcal{R}(y)$$

- The main difference is that this time we cannot add any correction terms to make our model exactly solvable. So we will continue perturbatively.
- This model has Schwarzschild solution as classical solution. We continue by finding the correction to this solution in the first order with respect to $\hbar$.

Eternal black hole

- Using same methods that we discussed in the case of eternal black hole in the BPP model, we get the following results:
- In the case without an island (at late times): $S_{FG} = \frac{N}{3}\kappa t$
- Where κ is quantum corrected surfice gravity. We see that we once again get Hawking's result.
- In the case with an island (at late times): $S = 2S_{BH}$.
- Again we get the same result as in the BPP model.
- The island appears near the horizon at radius: $a = r_h + \frac{\varepsilon}{MG}$
- Where r_h is quantum corrected event horizont and ϵ is perturbation parameter that is proportional to $\hbar$.

Thank you for your attention!