Time reparametrization and integrability of Chaplygin systems

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Integrable Hamiltonian systems

The phase space $\mathbb{R}^{2n}(x, p)$. Hamiltonian equations:

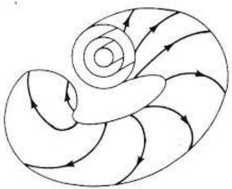
$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, \dots, n.$$

$f_1, \dots, f_{2n-d}$ 0 first integrals, such that

$$\{f_i, f_j\} = 0, \quad i = 1, \dots, d, \quad j = 1, \dots, 2n - d$$

Compact connected invariant level sets are d-dimensional tori. The trajectories are quasi-periodic:

$$\varphi_i(t) = \omega_i t + \varphi_{0i}, \quad i = 1, \dots, d.$$



The Neumann system

The motion of a point on the unit sphere
$S^{n-1} = \{\langle x, x\rangle = 1\} \subset \mathbb{R}^n$, with the quadratic potential

$$V(x) = \frac{1}{2}\langle x, Ax\rangle, \quad A = \mathrm{diag}(a_1, \dots, a_n) :$$

Tangent bundle TS^{n-1}: $\langle x, x\rangle = 1$, $\langle x, \dot{x}\rangle = 0$.

The equations:

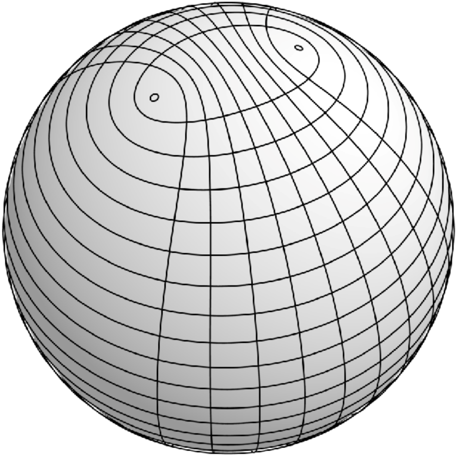
$$\ddot{x} = -Ax + \nu x,$$

where the Lagrange multiplier is $\nu = -\langle \dot{x}, \dot{x}\rangle + \langle x, Ax\rangle$.

The Hamilton-Jacobi equations are separable in sphero-conical variables and system is completely integrable.

Confocal cones and sphero-conical coordinates

$$Q_0(\lambda) = \left\{ \frac{x_1^2}{a_1 - \lambda} + \cdots + \frac{x_n^2}{a_n - \lambda} = 0 \right\}.$$



Confocal quadrics and elliptic coordinates

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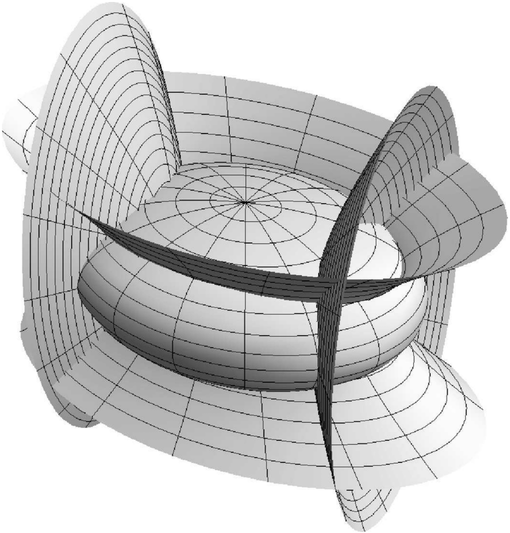

Geometric manifestation of integrability

Theorem (Moser)

Let $x(t)$ be a solution of the Neumann system on TS^{n-1}. Then the associated line

$$l(t) = \dot{x}(t) + \mathrm{span}\{x(t)\}$$

is tangent to $n-1$ fixed confocal quadrics of the family $Q_1(\lambda)$.

Theorem (Chasles)

Let $x(t)$ be a geodesic line on the ellipsoid

$$E^{n-1} = Q_1(0) = \langle x, A^{-1}x \rangle = 1,$$

then the associated line

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Nonholonomic systems

We consider $(M, L, \mathcal{D})$ a nonholonomic Lagrangian system, where $\mathcal{D}$, is locally defined by 1-forms α^a, $a = 1, \dots, k$

$$(\alpha^a, \dot{q}) = \sum_{i=1}^{n} \alpha_i^a(q)\dot{q}^i = 0, \qquad a = 1, \dots, k.$$

The equations of the motions are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} + \sum_{a=1}^{k} \lambda_a \alpha_i^a, \qquad i = 1, \dots, n, \qquad \sum_{i=1}^{n} \alpha_i^a(q)\dot{q}^i = 0$$

For natural mechanical systems $L = \frac{1}{2}\langle \dot{q}, \dot{q} \rangle = \frac{1}{2}\sum_{ij} g_{ij}\dot{q}^i\dot{q}^j - V(q)$ the equations become

$$\langle \nabla_{\dot{q}}\, \dot{q} + \mathrm{grad}\, V(q), \xi \rangle = 0, \qquad \dot{q}, \xi \in \mathcal{D}_q.$$

where ∇ is Levi-Chivita connection for the metric g.

It is natural to define *connection of the vector bundle* $\mathcal{D} \to M$:

$$\nabla^P \colon \Gamma(TM) \times \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D}), \qquad \nabla^P_X Y := P(\nabla_X Y),$$

where P is the orthogonal projection to $\mathcal{D}$. It is a metric connection and the equations are equivalent to

$$\nabla^P_{\dot{q}} \dot{q} = \mathrm{grad}_{\mathcal{D}} V(q), \qquad \dot{q} \in \mathcal{D}_q,$$

where $\mathrm{grad}_{\mathcal{D}} V = P(\mathrm{grad}\, V)$.
If $V \equiv 0$, one gets the equations of geodesic lines (so called *nonholonomic geodesics*)

$$\nabla^P_{\dot{q}} \dot{q} = 0, \qquad \dot{q} \in \mathcal{D}_q.$$

Chaplygin reduction

Suppose that $\pi : M \to N = M/G$ is a principal bundle with respect to the *left* action of a Lie group G, $\mathcal{D}$ is a G-invariant distribution (collection of *horizontal spaces*) and $T_qM = \mathcal{D}_q \oplus \mathcal{V}_q$ for all q, where $\mathcal{V}_q$ is tangent to the G–orbit through q. Given a vector $X_q \in T_qM$, there is a decomposition $X_q = X_q^h + X_q^v$. The *curvature* of $\mathcal{D}$ is the vertical valued 2-form K on M defined by

$$K(X_q, Y_q) = -[\bar{X}_q^h, \bar{Y}_q^h]_q^v,$$

where $\bar{X}$ and $\bar{Y}$ are smooth vector fields on M obtained by extending of X_q and Y_q.
Suppose that G acts by isometries on (M, g) and that V is G–invariant.

Then the equations are G-invariant and the restriction $L|_{\mathcal{D}}$ induces the reduced Lagrangian L_{red}, i.e, the reduced metric g_0 and the reduced potential energy V_0, via identification $TN \approx \mathcal{D}/G$. The *reduced Lagrange–d'Alambert* equations on the tangent bundle TN take the form

$$\Big(\frac{\partial L_{red}}{\partial x} - \frac{d}{dt}\frac{\partial L_{red}}{\partial \dot{x}}, \eta\Big) = \langle \dot{x}^h, K_q(\dot{x}^h, \eta^h)\rangle|_q \quad \text{for all} \quad \eta \in T_x N,$$

where $q \in \pi^{-1}(x)$ and $\dot{x}^h$ and η^h are unique horizontal lifts of $\dot{x}$ and η at q. The right-hand side can be written as $\Sigma(\dot{x}, \dot{x}, \eta)$ where Σ is $(0,3)$–tensor field on the base manifold N defined by

$$\Sigma_x(X, Y, Z) = \langle X^h, K_q(Y^h, Z^h)\rangle|_q, \qquad q \in \pi^{-1}(x).$$

The system $(M, g, V, \mathcal{D}, G)$ is referred to as *a G–Chaplygin system* (Koiller, Arch. R. Mech, 1992).

Let ∇^0 is the Levi-Civita connection of the reduced metric g_0. The reduced equations can be written as

$$\langle \nabla^0_{\dot{x}} \dot{x} + \operatorname{grad} V_0(x), \eta \rangle_0 + \Sigma(\dot{x}, \dot{x}, \eta) = 0,$$

that is

$$\nabla^0_{\dot{x}} + B(\dot{x}, \dot{x}) = -\operatorname{grad} V_0(x),$$

where the gradient is taken with respect to the reduced metric g_0, and tensor field B is defined by

$$\langle B(X, Y), Z \rangle_0 = \Sigma(X, Y, Z).$$

Now, the equations can be written

$$\nabla^B_{\dot{x}} \dot{x} = -\operatorname{grad} V_0,$$

where ∇^B is a symmetric connection defined by

$$\nabla^B_X Y = \nabla^0_X Y + \frac{1}{2}\big(B(X, Y) + B(Y, X)\big).$$

The connection ∇^B, for Abelian Chaplygin systems, is firstly introduced by Aleksandar Bakša (Mat. Vesnik, 1975)

Chaplygin multiplier method

The nonholonomic equations are not variational. For the reduced Abelian Chaplygin systems, Chaplygin proposed the Hamiltonization method using a time reparametrization $d\tau = \nu(x)dt$ now referred as a *Chaplygin multiplier*.
Denote $x' = dx/d\tau = \nu^{-1}\dot{x}$. The Lagrangian L_{red} in the coordinates (x, x') takes the form

$$L^*(x, x') = \frac{1}{2}\sum \nu^2 g_{0ij} x_i' x_j' - V_0(x).$$

We are looking for a function $\nu(x) \neq 0$ such that the reduced Chaplygin system

$$\frac{d}{dt}\frac{\partial L_{red}}{\partial \dot{x}_i} = \frac{\partial L_{red}}{\partial x_i} + \sum_{k,l,j=1}^{n} C_{ij}^k(x) g_{0kl}\dot{x}_l\dot{x}_j \qquad (1)$$

after a time reparametrization $d\tau = \nu(x)dt$ becomes the Lagrangian system

$$\frac{d}{dt}\frac{\partial L^*}{\partial x_i'} = \frac{\partial L^*}{\partial x_i}, \quad i = 1, \dots, n. \qquad (2)$$

In terms of connections, the Chaplygin multiplier is a function $\nu(x) \neq 0$ such that the reduced equation in the new time takes the form

$$\nabla^*_{x'} x' = - \operatorname{grad}_* V_0,$$

where ∇^* is the Levi-Civita connection of the conformal metric $g_* = \nu^2 g_0$ on the base manifold N.

Time reparametrization and conformal metrics

We will slightly modify the Chaplygin method by allowing the conformal factor and multiplier ν to be independent.

Consider the conformal metrics $g_* = f^2 g$ and g on M ($f \neq 0$ on M). The coefficients of their Levi-Civita connections ∇^* and ∇ are related by

$$\Gamma^{*k}_{ij} = \Gamma^k_{ij} + \frac{1}{f}\big(\delta^k_j \frac{\partial f}{\partial q^i} + \delta^k_i \frac{\partial f}{\partial q^j} - g_{ij} g^{kl} \frac{\partial f}{\partial q^l}\big).$$

Consider the geodesic equations $\nabla^*_{q'} q' = 0$ of (M, g_*),

$$\frac{d^2 q^k}{d\tau^2} + \Gamma^{*k}_{ij} \frac{dq^i}{d\tau}\frac{dq^j}{d\tau} = 0,$$

with respect to the affine parameter τ, and perform the time-reparametrisation

$$d\tau = \nu(q) dt : \quad \dot{q} = \nu \cdot q' \qquad (\nu \neq 0).$$

The equations can be written as

$$\ddot{q}^k + \Gamma^k_{ij}\dot{q}^i\dot{q}^j = \frac{\partial \ln \nu}{\partial q^r}\dot{q}^r\dot{q}^k - \frac{1}{f}\big(2\frac{\partial f}{\partial q^i}\dot{q}^i\dot{q}^k - g_{ij}g^{kl}\frac{\partial f}{\partial q^l}\dot{q}^i\dot{q}^j\big).$$

Proposition

Assume that on a Riemannian manifold (M, g) we have Newton equations

$$\nabla_{\dot{q}} \dot{q} = F(\dot{q}, q),$$

such that the force field can be written in the form

$$F = \langle \operatorname{grad} \ln \nu, \dot{q} \rangle \dot{q} - 2 \langle \operatorname{grad} \ln f, \dot{q} \rangle \dot{q} + \langle \dot{q}, \dot{q} \rangle \operatorname{grad} \ln f,$$

for certain functions $f, \nu \neq 0$, on M. Then, after a time reparametrisation $d\tau = \nu(q) dt$, the equations take the form of the equations of the geodesic lines

$$\nabla^*_{q'} q' = 0$$

of the metric $g_* = f^2 g$.

If we take $\nu = f^\alpha$, the above expression is slightly simplified:

$$F = (\alpha - 2) \langle \operatorname{grad} \ln f, \dot{q} \rangle \dot{q} + \langle \dot{q}, \dot{q} \rangle \operatorname{grad} \ln f.$$

Note that the geodesic equation $\nabla^*_{q'} q' = 0$ has the kinetic energy integral $\frac{1}{2}\langle q', q'\rangle_*$. Therefore, the system $\nabla_{\dot q}\, \dot q = F(\dot q, q)$, has the quadratic first integral $f^2/2\nu^2\langle \dot q, \dot q\rangle$, which is an obstruction to the construction.
However, Proposition can be formulated also with a weaker assumption: for the Newton equation having an invariant relation

$$\mathcal{E} = \Big\{(\dot q, q) \in TM \,|\, \frac{1}{2}\langle \dot q, \dot q\rangle - \frac{\nu^2}{f^2} = 0\Big\},$$

when the force F restricted to $\mathcal{E}$ reads

$$F = (\alpha - 2)\langle \operatorname{grad} \ln f, \dot q\rangle \dot q + f^{2\alpha-4} \operatorname{grad} f^2, \quad \text{for} \quad \nu = f^\alpha.$$

Then the solution of $\nabla_{\dot q}\, \dot q = F(\dot q, q)$ that belong to the invariant surface $\mathcal{E}$ are mapped to the geodesic lines $\nabla^*_{q'} q' = 0$ with the unit kinetic energy $\frac{1}{2}\langle q', q'\rangle_* = 1$.

In the case $\alpha = 2$, we have $F = \text{grad}\, f^2$. By taking $f = \sqrt{h - V(q)}$, the invariant relation is

$$\frac{1}{2}\langle \dot{q}, \dot{q} \rangle + V(q) = h,$$

and $F = -\,\text{grad}\, V$. We have the identity

$$\nabla^J_{q'}\, q' = (h - V)^{-2}\big(\nabla_{\dot{q}}\, \dot{q} + \text{grad}\, V\big),$$

where ∇^J is the Levi-Civita connection of the *Jacobi metric* $g_J = (h - V)g$ and

$$d\tau = (h - V)dt.$$

We obtain a well known formulation of the *Maupertuis principle*: the solutions $q(t)$ of the Newton equations $\nabla_{\dot{q}}\, \dot{q} = -\,\text{grad}\, V$ that satisfy $\frac{1}{2}\langle \dot{q}, \dot{q} \rangle + V(q) = h$, in the new time τ are geodesic lines $q(\tau)$ of the Jacobi metric with the unit kinetic energy $\frac{1}{2} g_J(q', q') = 1$.

Chaplygin ball rolling over the sphere in $\mathbb{R}^n$

We consider the Chaplygin ball type problem of rolling without slipping and twisting of an n-dimensional balanced ball of radius ρ in the following cases:

(i) rolling over outer surface of the $(n-1)$-dimensional fixed sphere of radius σ;

(ii) rolling over inner surface of the $(n-1)$-dimensional fixed sphere of radius σ ($\sigma > \rho$);

(iii) rolling over outer surface of the $(n-1)$-dimensional fixed sphere of radius σ, but the fixed sphere is within the rolling ball ($\sigma < \rho$, in this case, the rolling ball is actually a spherical shell).

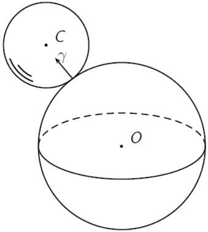

Fig 1a: Rolling over sphere

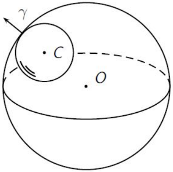

Fig 1b: Rolling within sphere

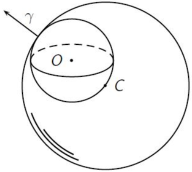

Fig 1c: Rolling shell over fixed sphere placed inside

Configuration space is direct product of Lie groups $SO(n)$ and $\mathbb{R}^n$. $g \in SO(n)$ is the rotation matrix, which maps a frame attached to the body to the space frame, and $\mathbf{r} = \overrightarrow{OC} \in \mathbb{R}^n$ is the position vector of the ball center C in the space frame, where the origin O coincides with the center of the fixed sphere. The vector $\mathbf{r}$ belongs to the $(n-1)$-dimensional constraint sphere defined by $(\mathbf{r}, \mathbf{r}) = (\sigma \pm \rho)^2$ ("$+$"for the case (i) and "$-$"for the cases (ii) and (iii)).

The condition that the ball to role without slipping and the of non-twisting at the contact point defines $(n-1)$-dimensional constraint distribution $\mathcal{D}$, which is the principal connection of the bundle

$$SO(n) \longrightarrow SO(n) \times S^{n-1} \xrightarrow{\pi} S^{n-1}$$

with respect to the $SO(n)$-action $a \cdot (g, \mathbf{r}) = (ag, a\mathbf{r})$, $a \in SO(n)$. The submersion π is given by

$$\gamma = \pi(g, \mathbf{r}) = \frac{1}{\sigma \pm \rho} g^{-1} \mathbf{r}$$

and γ is a unit vector, the direction of the contact point in the frame attached to the ball. Thus, the problem of the rubber rolling of a ball over a fixed sphere is a $SO(n)$–Chaplygin system and reduces to the tangent bundle $TS^{n-1} \cong \mathcal{D}/SO(n)$.

The equation describing the motion of the reduced system are

$$\Big(\epsilon \frac{d}{dt}\big(\mathbf{I}(\gamma \wedge \dot\gamma)\gamma\big) + (1-\epsilon)\mathbf{I}(\gamma \wedge \dot\gamma)\dot\gamma, \xi\Big) = 0, \qquad \xi \in T_\gamma S^{n-1}.$$

where $\mathbf{I} = \mathbb{I} + D \cdot \mathrm{Id}_{so(n)}$, $D = m\rho^2$, and $\epsilon = \sigma/(\sigma \pm \rho)$.
Let $A = \mathrm{diag}(a_1, \ldots, a_n)$. For the special inertia operator

$$\mathbb{I}(E_i \wedge E_j) = (a_i a_j - D)E_i \wedge E_j \quad \text{i.e.,} \quad \mathbf{I}(X \wedge Y) = AX \wedge AY.$$

under the time substitution $d\tau = \epsilon (A\gamma, \gamma)^{\frac{1}{2\epsilon}-1}\, dt$, the reduced system becomes the geodesic flow of the metric g_* with the Lagrangian

$$L^*(\gamma', \gamma) = \frac{1}{2}(\gamma, A\gamma)^{\frac{1}{\epsilon}-2}\big((A\gamma', \gamma')(A\gamma, \gamma) - (A\gamma, \gamma')^2\big)$$

Integrability for $\rho = 2\sigma$

In three-dimensional case, Borisov and Mamaev (RCD 2007) proved the integrability of the rubber rolling for a specific ratio between radiuses of the ball and the spherical shell (the case (iii), where $\rho = 2\sigma$, i.e, $\epsilon = -1$). We proceed in proving the complete integrability of the *n*-dimensional variant of the problem

Lemma

Under the transformation

$$x = \frac{A^{\frac{1}{2}}\gamma}{\sqrt{(A\gamma,\gamma)}}.$$

the metric g_* transforms to the metric

$$\mathbf{g}(X,Y) = (x, A^{-1}x)^{-\frac{1}{\epsilon}}(X,Y), \qquad X,Y \in T_x S^{n-1},$$

conformally equivalent to the standard metric on the sphere

$$(x,x) = 1.$$

Considered a natural mechanical system on the sphere $(x,x) = 1$ with the Lagrangian

$$L_\epsilon = \frac{1}{2}\left(\frac{dx}{ds},\frac{dx}{ds}\right) - V_\epsilon(x), \quad V_\epsilon(x) = -(A^{-1}x,x)^{-\frac{1}{\epsilon}}.$$

According to the Maupertuis principle, the trajectories $x(s)$ of the system with Lagrangian L_ϵ laying on the zero-energy invariant surface

$$\frac{1}{2}(\frac{dx}{ds}, \frac{dx}{ds}) - (A^{-1}x, x)^{-\frac{1}{\epsilon}} = 0, \qquad (3)$$

after a time reparametrization

$$d\tau = (A^{-1}x, x)^{-\frac{1}{\epsilon}} ds,$$

become the geodesic lines $x(\tau)$ of the metric $\mathbf{g}$ with the unit kinetic energy $\frac{1}{2}\mathbf{g}(x', x') = 1$ $(x' = dx/d\tau)$.

On the other hand, the solutions $\gamma(t)$ of the reduced nonholonomic problem, after a time reparametrization

$$d\tau = \epsilon(A\gamma, \gamma)^{\frac{1}{2\epsilon}-1} dt = \epsilon(A^{-1}x, x)^{1-\frac{1}{2\epsilon}} dt$$

become the geodesic lines $x(\tau)$ of the metric $\mathbf{g}$ with the same kinetic energy

$$\frac{1}{2}g_0(\dot\gamma, \dot\gamma) = \frac{1}{2}g_*(\gamma', \gamma') = \frac{1}{2}\mathbf{g}(x', x').$$

Combining the above transformations we obtain the proof of the statement.

Proposition

The trajectories $\gamma(t)$ of the rolling of a rubber Chaplygin ball over a spherical surface with the unit velocity $\langle\dot\gamma,\dot\gamma\rangle_0 = 1$, under the transformation $x = \frac{A^{\frac{1}{2}}\gamma}{\sqrt{(A\gamma,\gamma)}}$ and time reparametrisation

$$ds = \epsilon(A^{-1}x, x)^{1+\frac{1}{2\epsilon}}dt \quad (= \epsilon(A\gamma,\gamma)^{-1-\frac{1}{2\epsilon}}dt),$$

are mapped to the zero-energy trajectories $x(s)$ of the natural mechanical systems with the Lagrangian L_ϵ:

$$\frac{d^2}{ds^2}x = -\frac{2}{\epsilon}\big(A^{-1}x, x\big)^{-\frac{1}{\epsilon}-1}A^{-1}x + \lambda x, \quad \lambda = \frac{2}{\epsilon}\big(A^{-1}x, x\big)^{-\frac{1}{\epsilon}} - \Big(\frac{dx}{ds}, \frac{dx}{ds}\Big).$$

Among the potentials V_ϵ, there are two exceptional ones determining completely integrable systems: for $\epsilon = +1$ we have Braden's and for $\epsilon = -1$ Neumann's potential.

Theorem

For an special inertia operator and $\rho = 2\sigma$ ($\epsilon = -1$), the reduced problem of the rolling of a rubber Chaplygin ball over a spherical surface is completely integrable: under time reparametrisation

$$ds = -(A^{-1}x, x)^{\frac{1}{2}} dt \quad (= -(A\gamma, \gamma)^{-\frac{1}{2}} dt),$$

the solutions $\gamma(t)$ of reduced equation with the unit velocity $\langle \dot\gamma, \dot\gamma \rangle_0 = 1$ are mapped to the zero-energy trajectories $x(s)$ of the Neumann system with Lagrangian L_{-1}.

Symmetric ball

Theorem

For the inertia operator $\mathbb{I}(E_i \wedge E_j) = (a_i a_j - D) E_i \wedge E_j$, *where*

$$a_1 = a_2 = \cdots = a_l = \alpha_0 \neq a_{l+1} = a_{l+2} = \cdots = a_n = \alpha_1,$$

the reduced system is integrable for all ϵ*: generic motions, up to a time reparametrisation, are quasi periodic over three dimensional invariant tori. For* $l = 1$ *or* $l = n - 1$*, the invariant tori are two-dimensional.*

Almost Hamiltonian formulation of Chaplygin systems

The Hamiltonian function

$$H(x,p) = \frac{1}{2}(p, g^{-1}(p)) + v(x) = \frac{1}{2}\sum g^{0ij} p_i p_j + V_0(x)$$

(the usual Legendre transformation of L_{red}), where $(p_1, \ldots, p_n, x_1, \ldots, x_n)$ are the canonical coordinates of the catangent bundle T^*N,

$$p_i = \partial L_{red}/\partial \dot{x}_i = \sum_j g_{ij}\dot{x}_j,$$

and $\{g^{0ij}\}$ is the inverse of the metric matrix $\{g_{0ij}\}$. In the canonical coordinates the reduced equations take the form

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \sum_{j=1}^{n} g^{ij} p_j, \qquad (4)$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} + \sum_{k,j=1}^{n} C_{ij}^{k}(x) p_k \frac{\partial H}{\partial p_j}. \qquad (5)$$

Chaplygin multiplier: from the Lagrangian to the Hamiltonian framework

Consider the time substitution $d\tau = \nu(x)dt$ and the Lagrangian function $L^*(x, x')$. Then the conjugate momenta are

$$\tilde{p}_i = \partial L^* / \partial x'_i = \nu^2 \sum_j g_{0ij} x'_j,$$

and the corresponding Hamiltonian reads

$$H^*(x, \tilde{p}) = \frac{1}{2} \sum \frac{1}{\nu^2} g^{0ij} \tilde{p}_i \tilde{p}_j + V_0(x).$$

We have the following commutative diagram:

$$\begin{array}{ccc} TN\{x, \dot{x}\} & \xrightarrow{x' = \nu^{-1}\dot{x}} & TN\{x, x'\} \\ {\scriptstyle p = g_0(\dot{x})}\downarrow & & \downarrow{\scriptstyle \tilde{p} = \nu^2 g_0(x')} \\ T^*N\{x, p\} & \xrightarrow{\tilde{p} = \nu p} & T^*N\{x, \tilde{p}\}. \end{array}$$

Let $\tilde{\Omega}$ be the canonical symplectic form on T^*N with respect to the coordinates $(x, \tilde{p})$. Then

$$\tilde{\Omega} = \sum_i d\tilde{p}_i \wedge dx_i = \nu\Omega + d\nu \wedge \theta, \quad \theta = p_1 dx_1 + \dots p_n dx_n, \quad \Omega = d\theta.$$

Thus, H and H^* represents the same Hamiltonian function on T^*N written in two coordinate systems related by non-canonical change of variables

$$(x, p) \longmapsto (x, \tilde{p}). \tag{6}$$

The function ν is the *Chaplygin multiplier* if the equations (4), (5), after the time parametrisation $d\tau = \nu(x)dt$ and coordinate transformation (6) becomes the Hamiltonian equation with respect to the symplectic form $\tilde{\Omega}$, that is, we have:

$$x_i' = \frac{\partial H^*}{\partial \tilde{p}_i}(x, \tilde{p}), \qquad \tilde{p}_i' = -\frac{\partial H^*}{\partial x_i}(x, \tilde{p}).$$

Chaplygin multiplier and an invariant measure

If ν is Chaplygin multiplier, then the reduced equations

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \sum_{j=1}^{n} g^{ij} p_j,$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} + \sum_{k,j=1}^{n} C_{ij}^{k}(x) p_k \frac{\partial H}{\partial p_j}.$$

preserve the measure $\nu^{n-1}\Omega^n$. For $n = 2$ the statement can be inverted.

Manakov metrics

Generally, for $n \geq 4$, the operator $\mathbb{I}(E_i \wedge E_j) = (a_i a_j - D) E_i \wedge E_j$ is not a physical inertia operator of a multidimensional rigid body that has the form

$$\omega \longmapsto I\omega + \omega I, \qquad I = \operatorname{diag}(I_1, \ldots, I_n).$$

Here I is a positive definite matrix called the *mass tensor*, which is diagonal in the moving orthonormal base determined by the principal axes of inertia.

Then we can write the modified operator $\mathbf{I} = \mathbb{I} + D \cdot \mathrm{Id}_{so(n)}$ as

$$\mathbf{I}\,\omega = \epsilon^2 \big(J\omega + \omega J \big),$$

where

$$J = \operatorname{diag}(J_1, \ldots, J_n) = \frac{1}{\epsilon^2} \operatorname{diag}\Big(I_1 + \frac{D}{2}, \cdots, I_n + \frac{D}{2} \Big),$$

and the Legendre transformation takes the form

$$p = -J(\gamma \wedge \dot{\gamma})\gamma - (\gamma \wedge \dot{\gamma}) J\gamma = J\dot{\gamma} + (J\gamma, \gamma)\dot{\gamma} - (J\gamma, \dot{\gamma})\gamma.$$

The reduced equations

The point (p, γ) belongs to the cotangent bundle of a sphere realized as a symplectic submanifold in the symplectic linear space $(\mathbb{R}^{2n}(p, \gamma), dp_1 \wedge d\gamma_1 + \cdots + dp_n \wedge d\gamma_n)$:

$$(\gamma, \gamma) = 1, \qquad (\gamma, p) = 0.$$

The reduced equations:

$$\dot{\gamma} = C_\gamma\big(p - \frac{(p, C_\gamma(\gamma))}{(\gamma, C_\gamma(\gamma))}\gamma\big) \tag{7}$$

$$\dot{p} = \frac{1-\epsilon}{\epsilon}\big((X_\gamma, X_\gamma)J\gamma - (\gamma, JX_\gamma)X_\gamma - (X_\gamma, X_\gamma)(\gamma, J\gamma)\gamma\big) - 2H\gamma, \tag{8}$$

Here

$$C_\gamma = \mathrm{diag}(J_1 + (\gamma, J\gamma), \cdots, J_n + (\gamma, J\gamma))^{-1},$$

$$H(p, \gamma) = \frac{1}{2}(p, C_\gamma(p)) - \frac{1}{2}\frac{(p, C_\gamma(\gamma))^2}{(\gamma, C_\gamma(\gamma))}.$$

Invariant measure

Let $\mathbf{w}$ be the canonical symplectic form on T^*S^{n-1}:

$$\mathbf{w} = dp_1 \wedge d\gamma_1 + \cdots + dp_n \wedge d\gamma_n \,|_{T^*S^{n-1}}.$$

Theorem

The reduced system (7), (8) *has an invariant measure*

$$\mu(\gamma)\mathbf{w}^{n-1} = \left(\frac{\det C_\gamma}{(\gamma, C_\gamma(\gamma))}\right)^{1-\frac{1}{2\epsilon}} \mathbf{w}^{n-1}. \qquad (9)$$

Theorem

Assume that $n \geq 4$, $\epsilon \neq 1/2$, and $J_i = J_j \neq J_k = J_l$ for some mutually different indexes i, j, k, l. Then the reduced flow does not allow a Chaplygin multiplier.

Integrability of a symmetric case

Note that in the case of $SO(n)$–symmetry, when the mass tensor I (i.e., the matrix J) is proportional to the identity matrix, the $(0,3)$-tensor Σ vanishes and the trajectories are great circles for all ϵ.

Theorem

For the symmetric inertia operator $I(\omega) = I\omega + \omega I$, $I = \mathrm{diag}(I_1, \dots, I_n)$,

$$I_1 = I_2 = \cdots = I_r \neq I_{r+1} = I_{r+2} = \cdots = I_n$$

the reduced system (7), (8) *is solvable by quadratures and we have:*
(i) If $r \neq 1, n-1$*, generic motions are quasi-periodic over 3–dimensional invariant tori that are level sets of integrals* H, ϕ_{ij}, ϕ_{kl}, $1 \leq j < i \leq r$, $r < l < k \leq n$.
(ii) If $r = n-1$*, generic motions are quasi-periodic over 2–dimensional invariant tori that are level sets of* H, ϕ_{ij}, $1 \leq j < i \leq n-1$ *(similarly for* $r = 1$*).*

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