Evolution of quantum correlations in Gaussian noisy channels

Aurelian Isar

Department of Theoretical Physics
National Institute of Physics and Nuclear Engineering
Bucharest-Magurele, Romania
isar@theory.nipne.ro

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Outline

- q. corrs (Bell nonlocality, entanglement, discord, steering) - important phys. resources for many q. information process. and commun. protocols and tasks
- huge boost in development of QIT - from a resource th. approach to entanglement, expansion of this framework to discord and general q. corrs
- nec. cond. for all types of q. corrs
- recently, a framework for the quantification of coherence has been established, in which q. coherence is treated as a resource in a manner similar to entanglement
- main results so far apply mostly to finite dim. setting
- in th. of OQSs based on completely pos. q. dyn. semigs - quantification of coherence in Gaussian states of CV ss
- description of q. coherence using rel. entropy as a measure for q. corrs between 2 bosonic modes in a squeezed therm. env.
- evaluate rel. entropy of coherence by simulating effects of a sq. therm. bath in time on the s - picture of deco in OQS

Quantum versus classical

- Bell nonlocality, entanglement, coherence, purity (resources for q. technological applications: q. communication, q. computation, q. metrology)
- q. technology
- q. thermodynamics
- The crucial difference between class. and q. physics: superposition principle
- celebrated ex. - Schrödinger cat: $a|alive > +d|dead >$
- class. ss - can only exist in class. states $|m >$
- q. ss - can exist in superpositions $|\psi >= \sum_i a_i|i >, \sum_i |a_i|^2 = 1$
- q. coherence - at least two nonzero coeffs a_i

Quantum resources

- Entanglement
- Coherence
- Purity
- Asymmetry
- Bell-nonlocality
- Contextuality
- Steering
- Non-Markovianity
- Non-Gaussianity
- Quantum Correlations
  E. Chitambar and G. Gour, Reviews of Modern Physics (2019)
- Aims of a resource theory: quantification of the resource, interconversion properties, classification of operations

Quantum resource theory

- resource th. approach to q. correlations (entanglement, discord, steering), q. coherence, q. purity
- 2 main ingredients to define a resource th: set of resource free states (F) (contain no resource - useless but provided without cost) (versus resource states - useful but costly) and set of resource free operations - q. transfs. which can be performed at no cost - cost nothing to implement
- free operations can map free states only to free states; Operation = CPTP map, i.e. $\Gamma(\rho) = \sum_i K_i \rho K_i^\dagger$ (choice of free operations is not unique)
- ex. - resource th. of entanglement: non-entangled (separable) states - set of free states; LOCC - set of free operations; resource operations: entanglement-creating
- in gen., a proper resource measure $\bar{D}$ is def. as a pseudo-distance D between the state ρ and the closest free state $\rho_F \in F$: $\bar{D}(\rho) = \inf_F D(\rho, \rho_F)$
  D is a pseudo-distance: semi-positive ($D(\rho) \geq 0$), contractive $D(\rho_1, \rho_2) \geq D(\Gamma(\rho_1), \Gamma(\rho_2))$, but not nec. symm.

Resource theory of quantum coherence

- concept of coherence is inherently basis dependent
- a state is incoherent (coherence free state) if it is diagonal when expressed in a fixed orthonormal basis $|i>$
  $\rho = \sum_i p_i |i><i|, p_i \geq 0, \sum_i p_i = 1$ (set of free states $\mathcal{I}$)
- free operations:
- incoherent operations (IO): q. measurements which cannot create coherence for post-measurement states
  $\Gamma_{\mathrm{IO}}[\rho] = \sum_i K_i \rho K_i^\dagger$, with incoherent Kraus operators K_i, i.e. $K_i|m> \sim |n>$
- maximally incoherent operations (MIO): most general set, contains all transfs which cannot create coherence
  $\Gamma_{\mathrm{MIO}}[\rho] \in \mathcal{I}, \forall \rho \in \mathcal{I}$
- genuinely incoherent operations (GIO): transfs which preserve incoherent states: $|i> \to |i>$ (capture the framework of coherence under energy preservation)
- thermal operations (TO): transfs which are compatible with the first and second law of thermodynamics; TO cannot create coherence and are genuinely incoherent

Resource states

- resource th. of entanglement: studies resources for establishing long-distance entanglement
  - resource state: maximally entangled state
  $|\phi^+ >= \frac{1}{\sqrt{2}}(|00> + |11>)$
  - entanglement distillation: asymptotic conversion
  $\rho \to |\phi^+ >$
- resource th. of coherence: studies resources for establishing superpositions
  - resource state: maximally coherent state
  $|+ >= \frac{1}{\sqrt{2}}(|0> + |1>)$
  - coherence distillation: asymptotic conversion $\rho \to |+ >$
- resource th. of purity: studies resources for creating pure q. states
  - free state: maximally mixed state $\frac{I}{d}$
  - free operations: unital operations $\Lambda_U[\frac{I}{d}] = \frac{I}{d}$
  - resource states: all pure states $|\psi >$
  - resource distillation: asymptotic conversion $\rho \to |\psi >$

Relative entropy

- quantification of the amount of coherence via coherence monotone C (non-increasing under free operations), i.e. $C(\Gamma_{\mathrm{MIO}}[\rho]) \leq C(\rho)$
- important example: distance-based coherence monotone: $C(\rho) = \inf_{\delta \in \mathcal{I}} D(\rho, \delta), \delta \in \mathcal{I}$, with D suitable distance, e.g. relative entropy of coherence: $C_r(\rho) = \inf_{\delta \in \mathcal{I}} S(\rho|\delta), \delta \in \mathcal{I}$ with $S(\rho|\delta) = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\rho \log \delta)$
- $S(\rho_1|\rho_2) = -S(\rho_1) - \mathrm{Tr}(\rho_1 \log \rho_2)$, $S(\rho)$ - von Neumann entropy $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$, q. analogue of Shannon entropy from class. information th.
- relative entropy is a pseudo-distance, in the sense that it is semi-positive and contractive, but not symmetric:

$$
\begin{aligned}
S(\rho_1|\rho_2) &\geq 0, \\
S(\rho_1|\rho_2) &\geq S(\Gamma(\rho_1)|\Gamma(\rho_2)), \\
S(\rho_1|\rho_2) &\neq S(\rho_2|\rho_1)
\end{aligned}
$$

Γ- completely pos. trace preserving (CPTP) map

Relative entropy of coherence for CV ss

- relative entropy can be used to define a measure for q. coherence, namely relative entropy of coherence: relative entropy between a given Gaussian state and the closest Gaussian thermal state
- this is only an upper bound because the Gaussian set is not convex; ideally one should search for the closest incoherent quantum state, which in general is not of Gaussian form
  $C(\rho) = \inf_\delta S(\rho|\delta) = \inf_\delta[-S(\rho) - \mathrm{Tr}(\rho \log \delta)]$
  ($\delta = \rho_{diag}$ - Gaussian thermal state)
  $\sup_\delta \mathrm{Tr}(\rho \log \delta)$ gives $\delta = \rho_{diag}$, then
  $C(\rho) = S(\rho_{diag}) - S(\rho)$
- for two-mode Gaussian state (J. Xu, Phys. Rev. A **93**, 032111 (2016))
- for a s. of 2 coupled bosonic modes in a common thermal env.
- in gen. - non-monotonic decay

Open systems

- the simplest dynamics for an OS which describes an irreversible process: semigroup of transformations introducing a preferred direction in time (characteristics for dissipative processes)
- in GKLS axiomatic formalism of introducing dissipation in quantum mechanics, the usual von Neumann-Liouville eq. ruling the time evolution of closed q. ss is replaced by the following Markovian master eq. (GKLS) for the density operator $\rho(t)$ in the Schrödinger rep.:

$$\frac{d\Phi_t(\rho)}{dt} = L(\Phi_t(\rho))$$

- Φ_t - the dynamical semigroup describing the irreversible time evolution of the open system and L is the infinitesimal generator of Φ_t
- fundamental properties are fulfilled (positivity, unitarity, Hermiticity)

Markovian master equation

- in axiomatic formalism based on CP q. dyn. semigs, irreversible time evolution of an OS (that incorporates the dissipative and noisy effects due to the environment) is described by Kossakowski-Lindblad Markovian master eq. for the density operator (Schrödinger rep.)

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \frac{1}{2\hbar}\sum_j (2V_j\rho(t)V_j^\dagger - \{\rho(t), V_j^\dagger V_j\}_+)$$

- H - Hamiltonian of the q. OS
- $V_j, V_j^\dagger$ - operators defined on the Hilbert space of H (model the interaction of OS with envir.)

- preserve the Gaussian nature of the state during time evolution of the system (Gaussian lossy channel)

Operators

- q. dyn. semigs that preserve in time Gaussian form of the states: H - polyn. of second degree in coordinates x, y and momenta p_x, p_y of the 2 q. OS and V_j, $V_j^\dagger$ - polyns. of first degree in canonical observables ($j = 1, 2, 3, 4$):

$$V_j = a_{xj} p_x + a_{yj} p_y + b_{xj} x + b_{yj} y$$

- Hamiltonian of 2 identical coupled modes:

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{m\omega^2}{2}(x^2 + y^2)$$

- dyn. semig. implies positivity of the matrix formed by the scalar products of the vectors $\mathbf{a}_x, \mathbf{a}_y, \mathbf{b}_x, \mathbf{b}_y$ (their entries are the components $a_{xj}, a_{yj}, b_{xj}, b_{yj}$, resp.)

Equations of motion

- bimodal covariance matrix

$$\sigma(t) = \begin{pmatrix} \sigma_{xx} & \sigma_{xp_x} & \sigma_{xy} & \sigma_{xp_y} \\ \sigma_{xp_x} & \sigma_{p_xp_x} & \sigma_{yp_x} & \sigma_{p_xp_y} \\ \sigma_{xy} & \sigma_{yp_x} & \sigma_{yy} & \sigma_{yp_y} \\ \sigma_{xp_y} & \sigma_{p_xp_y} & \sigma_{yp_y} & \sigma_{p_yp_y} \end{pmatrix}$$

$$\frac{d\sigma}{dt} = Y\sigma + \sigma Y^{\mathrm{T}} + 2D, \quad Y = \begin{pmatrix} -\lambda & 1/m & 0 & 0 \\ -m\omega^2 & -\lambda & -q & 0 \\ 0 & 0 & -\lambda & 1/m \\ -q & 0 & -m\omega^2 & -\lambda \end{pmatrix}$$

D - matrix of diffusion coefficients

$$D = \begin{pmatrix} D_{xx} & D_{xp_x} & D_{xy} & D_{xp_y} \\ D_{xp_x} & D_{p_xp_x} & D_{yp_x} & D_{p_xp_y} \\ D_{xy} & D_{yp_x} & D_{yy} & D_{yp_y} \\ D_{xp_y} & D_{p_xp_y} & D_{yp_y} & D_{p_yp_y} \end{pmatrix}$$

Time-dependent solution

$$\sigma(t) = \begin{pmatrix} \sigma_{xx} & \sigma_{xp_x} & \sigma_{xy} & \sigma_{xp_y} \\ \sigma_{xp_x} & \sigma_{p_xp_x} & \sigma_{yp_x} & \sigma_{p_xp_y} \\ \sigma_{xy} & \sigma_{yp_x} & \sigma_{yy} & \sigma_{yp_y} \\ \sigma_{xp_y} & \sigma_{p_xp_y} & \sigma_{yp_y} & \sigma_{p_yp_y} \end{pmatrix}$$

$$\sigma(t) = M(t)(\sigma(0) - \sigma(\infty))M^{\mathrm{T}}(t) + \sigma(\infty),$$

$M(t) = \exp(tY)$, $\lim_{t\to\infty} M(t) = 0$ (Y must only have eigenvalues with negative real parts)

$$Y\sigma(\infty) + \sigma(\infty)Y^{\mathrm{T}} = -2D$$

Covariance matrix

Two-mode Gaussian state is entirely specified by its covariance matrix σ, which is a real, symmetric and positive matrix

$$\sigma = \begin{pmatrix} A & C \\ C^{\mathrm{T}} & B \end{pmatrix}$$

(A, B and C - 2×2 Hermitian matrices: A and B - symm. covariance matrices for individual reduced one-mode states, C contains correlations between modes)

- for environments inducing asymptotic Gibbs state ($k = 1$)
- then we have equal unimodal covariance matrices $A = B$ and symmetric entanglement matrix C
- Gaussian states with $\det C \geq 0$ are separable, but for $\det C < 0$, it may be possible that states are entangled

Asymptotic covariance matrix

- asymptotic product Gibbs state describing a thermal equilibrium of the two modes with the squeezed thermal bath; asymptotic covariance matrix - determined by only the parameters of the squeezed thermal reservoirs: $\sigma(\infty) = \bigoplus_{i=1,2} \sigma_i(\infty)$, where the asymptotic covariance matrix of each bath is (we set $\hbar$=1)

$$\sigma_i(\infty) = \begin{pmatrix} (1 + 2N_i + 2\Re[M_i])/\omega_i & 2\Im[M_i] \\ 2\Im[M_i] & (1 + 2N_i - 2\Re[M_i])\,\omega_i \end{pmatrix}, \tag{1}$$

$\Re[M_i]$, $\Im[M_i]$ - real and imaginary part

$$\begin{aligned} M_i &= -\left(2n_{th,i} + 1\right) \cosh R_i \sinh R_i \exp i\phi_i, \\ N_i &= n_{th,i}\left(\cosh^2 R_i + \sinh^2 R_i\right) + \sinh^2 R_i. \end{aligned} \tag{2}$$

$n_{th,i} = \frac{1}{2}\left(\coth\left(\frac{\omega_i}{2T_i}\right) - 1\right)$ - average # of thermal photons, T_i - temperatures, R_i, ϕ_i - squeezing parameters and squeezing phases ($k_B = 1$ and $\omega_i = 1$)

Relative entropy of coherence - for two-mode Gaussian states

$C(\rho) = S(\rho_{diag}) - S(\rho) = \sum_{i=1}^{2}[g(2N_i + 1) - g(2k_i)]$
$g(x) = \frac{x+1}{2}\log\frac{x+1}{2} - \frac{x-1}{2}\log\frac{x-1}{2}$
k_1 and k_2 - symplectic eigenvalues of covariance matrix
- covariance matrix for the closest thermal state in the two-mode case - described by the average nos of thermal photons N_1 and N_2
$N_1 = \frac{1}{2}(A_{11} + A_{22} - 1)$
$N_2 = \frac{1}{2}(B_{11} + B_{22} - 1)$

$$\sigma = \begin{pmatrix} A & C \\ C^{\mathrm{T}} & B \end{pmatrix}$$

$2k_1^2 = \Delta - \sqrt{\Delta^2 - 4\det\sigma}$
$2k_2^2 = \Delta + \sqrt{\Delta^2 - 4\det\sigma}$
$\Delta = \det A + \det B + 2\det C$ -symplectic invariant (seralian)

Entangled initial states

- initial Gaussian state: 2-mode STS, with CM (standard mode)

$$\sigma_{st}(0) = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ c & 0 & b & 0 \\ 0 & -c & 0 & b \end{pmatrix},$$

$$a = n_1 \cosh^2 r + n_2 \sinh^2 r + \frac{1}{2} \cosh 2r,$$

$$b = n_1 \sinh^2 r + n_2 \cosh^2 r + \frac{1}{2} \cosh 2r,$$

$$c = \frac{1}{2}(n_1 + n_2 + 1) \sinh 2r,$$

n_1, n_2 : average nos. of thermal photons; r : squeezing parameter; $n_1 = 0$ and $n_2 = 0 \rightarrow$ CM of the 2-mode SVS

- a 2-mode STS is entangled when the $r > r_s$, where

$$\cosh^2 r_s = \frac{(n_1+1)(n_2+1)}{n_1+n_2+1}$$

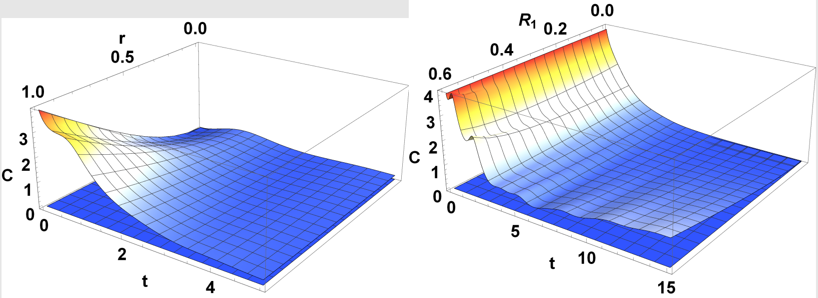

Figure: Relative entropy of coherence $C(t, r)$ (left) and $C(t, R_1)$ (right).

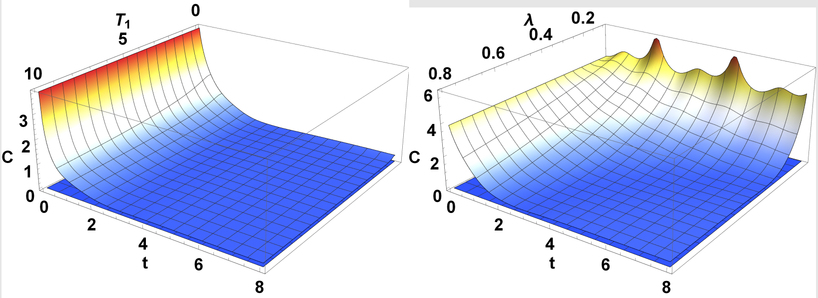

Figure: Relative entropy of coherence $C(t, T_1)$ (left) and $C(t, \lambda)$ (right).

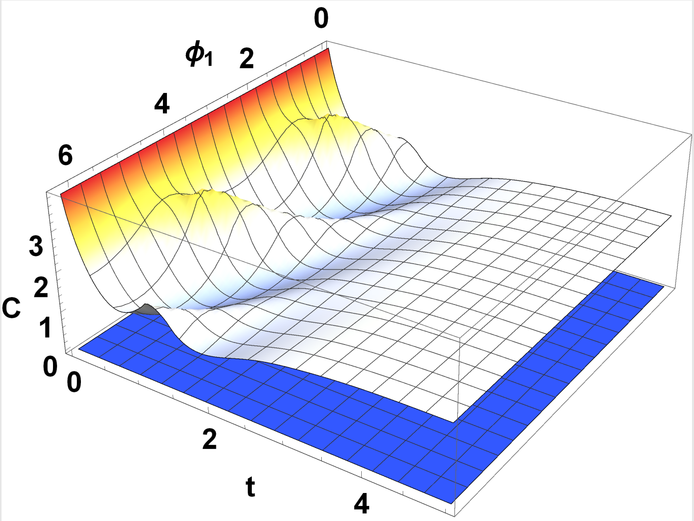

Figure: Relative entropy of coherence $C(t, \phi_1)$.

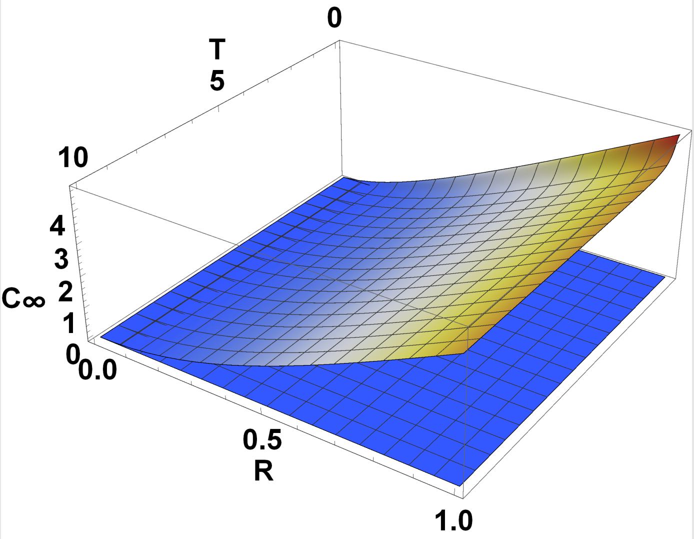

Figure: Relative entropy of coherence $C(T, R)$.

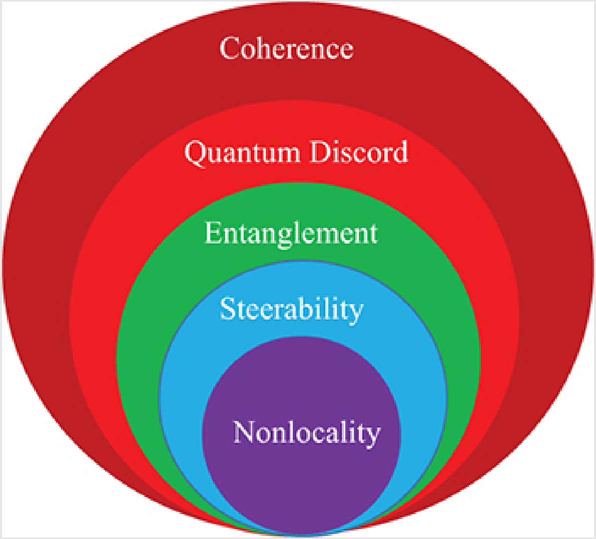

Figure: Hierarchy of correlations (V. Vedral)

Hierarchy of correlations

Hierarchy among quantum coherence and the various quantum correlations:
- in a basis-independent theory of quantum coherence, most gen. form of quantumness: coherence, and includes all the different forms of correlations such as discord, entanglement, steering and non-locality as subsets
- quantum dissonance corresponds to the set of all states with quantum discord, removing all entangled states

Conclusions

- relative entropy of coherence C for a system composed of two non-interacting modes embedded in a squeezed thermal bath, as a function of time, temperature, squeezing parameter of the modes and of env., average nos. of thermal photons
- it is possible to preserve indefinitely in time the quantum coherence of the system of two uncoupled Gaussian bosonic modes, interacting with the noisy environments, namely due to the squeezing of the reservoirs
- it is not strictly monotonous because the measure provides only an upper bound to coherence by only searching for the closest Gaussian thermal state and not a general quantum incoherent one

Summary and outlook

- quantum resource th.: strong framework for studying fundam. features of q. ss
- entanglement, coherence, and purity: resources for applications based on q. technology (q. communication, computation, metrology)
- relation between entanglement, coherence, purity
- role of coherence and entanglement in q. thermodynamics $\rightarrow$ understanding fundamental limitations for thermodyn. processes at small scales

Thank You!